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ON SOME COMBINATORIAL PROBLEMS ARISING IN PROBABILITY THEORY AND STATISTICS

by

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ABSTRACT

A considerable volume of recent research work has been devoted to the fluctuation theory of random variables and queueing theory. The purpose of this thesis is to study a very special class of combinatorial problems connected with these fields. Such problems are sometimes posed in the literature as generalized ballot theorems.

In Chapter I we review some recent work in these fields, namely, the work of Takács, Graham and Dwass. In Chapter II we generalize the ballot problem in yet another direction and obtain certain refinements of it using an analogue of the multinomial theorem. However, we have not been able to establish any connection between our work and the theorems of Chapter I though they appear to be somewhat similar. Chapter III contains a variety of results which are obtained as a by-product of our approach.

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CHAPTER I

A REVIEW OF CERTAIN COMBINATORIAL RESULTS IN QUEUEING THEORY AND FLUCTUATION THEORY

§ 1.1 Introduction.

In recent years a flood of books and research papers devoted to queueing theory and fluctuation theory has appeared. Thomas L. Saaty, in his book "Elements of queueing theory" published in 1961, lists over 900 research papers devoted to queueing theory alone. Professor Samuel Karlin reviewing [Mathematical Reviews, 3704 Vol. 24 Pt.A, 1962] L. Takács' "Introduction to the theory of queues" suggests that the subject 'fluctuation theory of random variables' originated in the analytic approach to certain problems in the theory of queues.

In 1951, E. Sparre Andersen's elementary combinatorial approach [1] provided a new fundamental insight regarding many problems of fluctuation theory. The celebrated work of Frank Spitzer [2] in 1956 on the distribution of the maximum of partial sums of independently identically distributed random variables laid the foundation for further work on the subject. However, not many authors have attempted to relate the work of Spitzer and others with queueing theory, although it is now well recognised that the two fields are somewhat connected. As one example of this connection, the busy time distribution for one server can be deduced from the work of Spitzer.

The purpose of this chapter is to review some research papers dealing with a class of combinatorial problems that have arisen independently

in the two fields. We propose to give some applications of these combinatorial problems leaving the interested reader to refer to Takács [3] for further applications, particularly to queueing theory.

The statements as well as the proofs of the theorems we review in the following sections will be of an elementary nature. However, it must be borne in mind that these results were suggested by Spitzer's work and arose through certain deep probability considerations. These theorems may, in fact, be considered a natural consequence of Spitzer's work.

§ 1.2 A review of some recent combinatorial results.

In this section we will review some combinatorial results which were first published by Takács in [3,4]. These results could also be obtained from fluctuation theory; and in section 1.4 we shall discuss more general results of this nature from an elementary point of view suggested by fluctuation theory. However, it was Takács who first gave several interesting applications of these results, and in this section we restrict ourselves to reviewing some of the results due to him [3,4].

Consider the following urn problem: Let an urn contain a cards each marked 0 and b cards each marked $\mu+1$. Suppose that all the $a+b$ cards are drawn without replacement from the urn. We seek the probability that for every $r = 1, 2, \ldots, a+b$ the sum of the first r numbers drawn is

(i) less than r, or

(ii) less than $(r+1)$.

We may ask equivalently: what is the probability that throughout the

drawing, the ratio of the number of zeros to the number of $(\mu+1)$'s is

(i) greater than $\mu : 1$, or

(ii) at least $\mu : 1$?

To see the equivalence of these two problems, let us suppose that amongst the first $r(=\alpha+\beta)$ draws there are α zeros and β $(\mu+1)$'s. Then

$$\alpha \cdot 0 + \beta \cdot (\mu+1) < \alpha + \beta$$

holds if and only if

$$\alpha > \mu\beta$$

and similarly

$$\alpha \cdot 0 + \beta \cdot (\mu+1) < \alpha + \beta + 1$$

holds if and only if

$$\alpha + 1 > \mu\beta$$

or, if and only if

$$\alpha \geqq \mu\beta \quad .$$

Having proved the equivalence of the above two problems we establish two simple theorems which we will need in solving the urn problem.

Theorem 1.2.1:

Consider an urn containing n cards marked with non-negative integers $k_1, k_2, \ldots, k_n$ respectively where $k_1 + k_2 + \ldots + k_n = k$ with $0 \leqq k \leqq n$.

Suppose that all the n cards are drawn without replacement from the urn. Let $\nu_j (j = 1, 2, \ldots, n)$ denote the number on the card drawn at the j^{th} draw. Then

(1.2.1) $$P\{\nu_1 + \nu_2 + \ldots + \nu_r < r \quad \text{for} \quad r = 1, 2, \ldots, n\} = 1 - \frac{k}{n}$$

irrespective of the particular values of $k_1, k_2, \ldots, k_n$.

Proof:

Let $\{\nu_1, \nu_2, \ldots, \nu_n\}$ be a permutation of $\{k_1, k_2, \ldots, k_n\}$. The random variables $\nu_1, \nu_2, \ldots, \nu_n$ are exchangeable i.e. for every r and $1 \leq i_1 < i_2 < \ldots < i_r \leq n$ the joint distribution of $\{\nu_{i_1}, \nu_{i_2}, \ldots, \nu_{i_r}\}$ agrees with that of $\{\nu_1, \nu_2, \ldots, \nu_r\}$. It follows that every permutation of $\{k_1, k_2, \ldots, k_n\}$ has the same probability of being chosen at random. Hence

$$\begin{aligned} E(\nu_i) &= \sum \nu_i \, p(\nu_i) \\ &= \sum \nu_i \cdot \frac{1}{n} \\ &= \frac{k}{n} \qquad\qquad (i = 1, 2, \ldots, n) \end{aligned}$$

We use induction to show that (1.2.1) holds for every pair (n,k) where $0 \leq k \leq n$.

When $n=1, k=0$, there is only one card marked zero so that

$$P(\nu_1 < 1) = 1$$

and (1.2.1) holds.

When $n=1$, $k=1$, $P(\nu_1 < 1) = 0$ because the only card is marked 1 and (1.2.1) still holds.

Hence (1.2.1) is true for $(1,0)$ and $(1,1)$. Assume that it holds for the pairs

$$(1,0),\ (1,1),\ \ldots,\ (n-1,0),\ \ldots,\ (n-1,n-1).$$

We shall prove that it also holds for the pair (n,k) where $0 \leqq k \leqq n$.

If $k = n$, (1.2.1) holds trivially. Let

$$\nu_1 + \nu_2 + \ldots + \nu_k = j \quad \text{with} \quad 0 \leqq j \leqq k .$$

Then

$$P\{\nu_1+\nu_2+ \ldots + \nu_r < r \quad \text{for} \quad r=1,\ 2,\ \ldots,\ n \mid \nu_1+\nu_2+ \ldots + \nu_k=j\}$$

$$= P\{\nu_1+\nu_2+ \ldots + \nu_r < r \quad \text{for} \quad r=1,\ 2,\ \ldots,\ k \mid \nu_1+\nu_2+ \ldots + \nu_k=j\}$$

$$= 1 - \frac{j}{k}$$

by the inductive hypothesis. Thus

$$P\{\nu_1 + \nu_2 + \ldots + \nu_r < r \quad \text{for} \quad r = 1,\ 2,\ \ldots,\ n\}$$

$$= \sum_{j=o}^{k} (1-\tfrac{j}{k})\ P\{\nu_1+\nu_2+ \ldots +\nu_k = j\}$$

$$= \sum_{j=o}^{k} P\{\nu_1+\nu_2+ \ldots +\nu_k=j\} - \frac{1}{k} \sum_{j=o}^{k} jP\{\nu_1+\nu_2+ \ldots +\nu_k=j\}$$

$$= 1 - \frac{1}{k} E(\nu_1 + \nu_2 + \ldots + \nu_k)$$

$$= 1 - \frac{1}{k} \{E(\nu_1) + E(\nu_2) + \ldots + E(\nu_k)\}$$

$$= 1 - \frac{1}{k} \quad \{k \cdot \frac{k}{n}\}$$

$$= 1 - \frac{k}{n} \quad .$$

Corollary:

Let $n_o, n_1, n_2, \ldots.$ denote the number of zeros, ones, twos, ... among $k_1, k_2, \ldots, k_n$ so that

$$0 \cdot n_o + 1 \cdot n_1 + 2 \cdot n_2 + \ldots. = \sum_i i\, n_i = k, \quad 0 \leq k \leq n \; .$$

Then under the asumption of Theorem 1.2.1 we have

$$(1.2.2) \quad P\{\nu_1+\nu_2+ \ldots + \nu_r < r+1 \quad \text{for} \quad r=1, 2, \ldots, n\} = \frac{(n+1-k)}{(n_o+1)} \; ,$$

$$(1.2.3) \quad P\{\nu_1+\nu_2+ \ldots + \nu_r < r+2 \quad \text{for} \quad r=1, 2, \ldots, n\} = \frac{(n+2-k)(n+1-n_1)}{(n_o+2)(n_o+1)} \; ,$$

$$(1.2.4) \quad P\{\nu_1+\nu_2+ \ldots +\nu_r < r+3 \quad \text{for} \quad r=1, 2, \ldots, n\}$$

$$= \frac{(n+3-k)[(n+1-n_1)(n+2-n_1)-n_2(n_o+3)]}{(n_o+3)(n_o+2)(n_o+1)}$$

etc.

Proof:

Let us introduce the notation

$$P_j(n,k,n_o,n_1, \ldots, n_{j-1}) = P\{\nu_1+\nu_2+ \ldots + \nu_r < r+j \quad \text{for} \quad r=1,2,\ldots,n\}$$

for $j=0, 1, 2, \ldots.$. These probabilities will be seen to be independent of $n_j, n_{j+1}, \ldots.$ and can be determined recursively from the relation

$$(1.2.5)\quad P_{j-1}(n+1,\ k,\ n_o+1,\ n_1,\ \dots,\ n_{j-2}) = \frac{(n_o+1)}{(n+1)} P_j(n,\ k,\ n_o,\ n_1,\dots,\ n_{j-1})$$
$$+ \sum_{i=1}^{j-1} \frac{n_i}{(n+1)} P_{j-i}(n,\ k-i,\ n_o+1,\ n_1,\ \dots,\ n_i-1,\ \dots,\ n_{j-i-1})$$

To prove (1.2.5) we proceed as follows:

Let us put into the urn a card marked zero so that now there are $(n+1)$ cards in all of which n_o+1 are marked zero and $n_i (i=1,2,\dots,k)$ are marked i. Then $P_{j-1}(n+1,\ k,\ n_o+1,\ n_1,\ \dots,\ n_{j-2})$ is the probability that throughout the drawing the sum of the first r numbers drawn is less than $r+j-1$.

If the first card drawn is marked zero, with a probability $\frac{(n_o+1)}{(n+1)}$, then there remain $(r-1)$ more cards to be drawn satisfying

$$\nu_2 + \nu_3 + \dots + \nu_r < r+j-1 \quad ,$$

the probability for which is $P_j(n,\ k,\ n_o,\ n_1,\ \dots,\ n_{j-1})$.

On the other hand, if the first card drawn is marked i $(i \neq 0,\ i = 1,\ 2,\ \dots,\ j-1)$ with a probability $\frac{n_i}{(n+1)}$ there remain $(r-1)$ more cards to be drawn out of n cards satisfying

$$\nu_2 + \nu_3 + \dots + \nu_r < r+j-i \quad .$$

The probability for this is $P_{j-i}(n,\ k-i,\ n_o+1,\ n_1,\ \dots,\ n_i-1,\ \dots\ n_{j-i-1})$. The events that the first card is marked $0,\ 1,\ \dots,\ j-1$ are mutually exclusive and (1.2.5) follows by the theorem of total probability.

In particular, $j=1$ gives

$$P_o(n+1,k) = \frac{(n_o+1)}{(n+1)} P_1(n, k, n_o) .$$

But $P_o(n+1, k)$ is the probability that the sum of the numbers on r cards is less than r for $r=1, 2, \ldots, n$. Thus by Theorem 1.2.1 we have

$$P_o(n+1,k) = 1 - \frac{k}{(n+1)}$$

giving

$$P_1(n,k,n_o) = \frac{(n+1)}{(n_o+1)} \cdot \frac{(n+1-k)}{(n+1)} = \frac{(n+1-k)}{(n_o+1)}$$

which proves (1.2.2) .

To prove (1.2.3) we let $j=2$ in (1.2.5) to get

$$P(n+1,k,n_o+1) = \frac{(n_o+1)}{(n+1)} P_2(n,k,n_o,n_1) + \frac{n_1}{(n+1)} P_1(n,k-1,n_o+1) .$$

Now replace n by $(n+1)$ and n_o by (n_o+1) in (1.2.2). This yields

$$P_1(n+1,k,n_o+1) = P_1(n,k-1,n_o+1) = \frac{(n+2-k)}{(n_o+2)}$$

so that

$$P_2(n,k,n_o,n_1) = \frac{(n+2-k)}{(n_o+2)} \cdot \left(1 - \frac{n_1}{(n+1)}\right) \cdot \frac{(n+1)}{(n_o+1)} ,$$

or,

$$P_2(n,k,n_o,n_1) = \frac{(n+2-k)(n+1-n_1)}{(n_o+1)(n_o+2)}$$

which proves (1.2.3).

Proceeding in a similar fashion we can obtain $P_j(n,k,n_o,\ldots,n_{j-1})$ for every j.

Theorem 1.2.2:

Let $\nu_1, \nu_2, \ldots, \nu_n$ be integral valued random variables and that all the n cyclic permutations of $(\nu_1, \nu_2, \ldots, \nu_n)$ have the same joint probability distribution. Let Δ_n denote the number of positive sums among $\nu_1 + \nu_2 + \ldots + \nu_r$ $(r=1, 2, \ldots, n)$. Then we have

(1.2.6) $$P\{\Delta_n = j \mid \nu_1+\nu_2+ \ldots + \nu_n = 1\} = \frac{1}{n} \qquad (j=1, 2, \ldots, n) .$$

Proof:

Let $k_1, k_2, \ldots, k_n$ be fixed integers with

$$k_1 + k_2 + \ldots + k_n = 1 .$$

Let us first suppose that $(\nu_1, \nu_2, \ldots, \nu_n)$ is being chosen at random among the n cyclic permutations of $(k_1, k_2, \ldots, k_n)$ and that each permutation has the same probability of being chosen. In this particular case we shall show that

$$P\{\Delta_n = j\} = 1/n \ , \qquad (j=1, \ldots, n)$$

irrespective of $k_1, k_2, \ldots, k_n$. Hence the theorem follows for the general case.

Consider n distinct numbers $n(k_1 + k_2 + \ldots + k_j)-j$ $(j=1,\ldots,n)$ and arrange them in an increasing order. Let i_j denote the serial number

of $n(k_1 + \dots + k_j)-j$.

Define $k_{i+n} = k_i$ $(i=1, \dots, n)$. We shall show that the cyclic permutation $(k_{j+1}, k_{j+2}, \dots, k_{j+n})$ of $(k_1, k_2, \dots, k_n)$ contains exactly $n+1-i_j$ positive partial sums, that is, the inequality

(1.2.7) $$k_{j+1} + k_{j+2} + \dots + k_r > 0 \qquad (r=j+1, j+2, \dots, j+n)$$

holds for <u>exactly</u> $(n+1-i_j)$ subscripts.

If $r=j+n$, then (1.2.7) holds, for

$$k_{j+1} + k_{j+2} + \dots + k_{j+n} = k_1 + k_2 + \dots + k_n = 1 > 0 .$$

If $r=j+1, j+2, \dots, j+n-1$ then (1.2.7) holds if and only if

(1.2.8) $$n(k_{j+1} + k_{j+2} + \dots + k_r) - (r-j) > 0 ,$$

that is, if

(1.2.9) $$n(k_1 + k_2 + \dots + k_r)-r > n(k_1 + k_2 + \dots + k_j)-j .$$

Now

$$\begin{aligned} n(k_1 + k_2 + \dots + k_{i+n}) - (i+n) &= n(k_1+k_2+ \dots +k_n+k_{1+n}+\dots+k_{1+i})-(i+n) \\ &= n(1 + k_1 + \dots + k_i)-(i+n) \\ &= n(k_1 + \dots + k_i) - i \end{aligned}$$

because $k_1 + k_2 + \dots + k_n = 1$ and $k_{j+n} = k_j$ $(j=1,2,\dots,n)$. Thus the inequality

(1.2.10) $$n(k_1+k_2+ \dots +k_r)-r > n(k_1+k_2+ \dots +k_j)-j \quad (r=1,2,\dots,n)$$

holds for the same number of values of r for which (1.2.8) holds with $r = j+1, j+2, \ldots, j+n-1$. By the definition of i_j, (1.2.10) holds for $n-i_j$ different values of r. Since $(n+1-i_j)$ $(j=1,2,\ldots,n)$ assumes each of the values $1, 2, \ldots, n$ only once, we can conclude that among the n cyclic permutations there is <u>exactly</u> one which has $j(j=1,2,\ldots,n)$ positive partial sums. This proves the theorem for these fixed values $(k_1,k_2,\ldots,k_n)$. Since the result is independent of $(k_1,k_2,\ldots,k_n)$, the theorem also holds in the general case.

We are now in a position to solve the urn problem. Let the urn contain $a+b$ cards, a of which are marked 0 and b marked $(\mu+1)$. In the notation of Theorem 1.2.1, we have

$$n = a+b \; , \quad k = (\mu+1)b \quad \text{and} \quad \nu_1+\nu_2+\ldots+\nu_r = \alpha\cdot 0+(\mu+1)\cdot\beta$$

where $r = \alpha+\beta$, α being the number of cards marked zero and β the number of cards marked $\mu+1$. Thus

$$P\{\nu_1+\nu_2+\ldots+\nu_r < r \quad \text{for} \quad r=1,2,\ldots,n\} = 1 - \frac{(\mu+1)b}{(a+b)}$$

i.e.

$$P\{0\cdot\alpha+(\mu+1)\cdot\beta < \alpha+\beta \quad \text{for} \quad \alpha+\beta = 1,2,\ldots,n\} = \frac{(a-\mu b)}{(a+b)}$$

or,

(1.2.11) $$P\{\alpha > \mu\beta\} = \frac{(a-\mu b)}{(a+b)} .$$

This gives a solution of the first part of the urn problem. To solve the second part we use the corollary to Theorem 1.2.1. In the above notation we have

$$P\{\nu_1+\nu_2+ \dots +\nu_r \leq r \quad \text{for} \quad r=1,\dots,n\}$$

$$= P\{\nu_1+\nu_2+ \dots +\nu_r < r+1 \quad \text{for} \quad r=1,\dots,n\}$$

$$= P\{0 \cdot \alpha + (\mu+1) \cdot \beta < \alpha+\beta+1\}$$

$$= P\{\alpha + 1 > \mu\beta\}$$

$$= P\{\alpha \geq \mu\beta\}$$

$$= \frac{a+b+1 - (\mu+1)b}{a+1}$$

or,

(1.2.12) $$P\{\alpha \geq \mu\beta\} = \frac{a+1-\mu b}{a+1} \quad .$$

We remark that in the next section we will interpret the urn problem as a generalised ballot problem and give yet another solution of the classical ballot problem.

§ 1.3 Applications of Takács theorems.

In this section we will give some applications of the theorems of section 1.2. The first application we wish to consider is to the classical ballot problems. Ballot theorems originated in 1887 as a mathematical puzzle. Many students of combinatorial analysis and probability theory have looked into the problem from different angles and obtained a variety of results. For historical remarks and references to further work we refer to Feller [5] and Takács [3,4].

Consider then the urn problem of Takács. Let us reformulate it as a generalised ballot problem as follows:

In a ballot candidate A scores a votes and candidate B scores b votes. Let $a \geq \mu b$ where $\mu \geq 0$ is arbitrary. We seek the probability that throughout the counting, the number of votes registered for A is

(i) always greater than μ times the number of votes registered for B , or

(ii) always at least μ times the number of votes registered for B .

We solve the ballot problem in its most general form. For a fixed $\mu \geq 0$ let us denote by $N(a,b)$ the number of ways of counting $a+b$ $(a > \mu b)$ votes such that throughout the counting, the number of votes for A is always greater than μ times the number of votes registered for B . Let $P(a,b)$ denote the corresponding probability. Then

$$P(a,b) = \frac{N(a,b)}{\binom{a+b}{a}}$$

and we have the

Theorem 1.3.1:

$$P(a,b) = \begin{cases} \dfrac{a}{a+b} \displaystyle\sum_{j=0}^{b} C_j \frac{\binom{b}{j}}{\binom{a+b-1}{j}} & \text{if } a > b\mu \\ 0 & \text{if } a \leq b\mu \end{cases} \tag{1.3.1}$$

where $C_o=1$ and the constants $C_j (j=1,2,\ldots)$ are given by the recurrence relation

$$\sum_{j=o}^{b} C_j \frac{\binom{b}{j}}{\binom{[b\mu]+b-1}{j}} = 0 \quad (b = 1,2,3,\ldots) \tag{1.3.2}$$

where $[b\mu]$ is the greatest integer $\leq b\mu$.

Proof:

If $a \leq b\mu$, then in every counting the number of votes for A cannot always be greater than μ times the number of votes for B throughout the counting and $N(a,b) = 0$. Thus $P(a,b) = 0$.

Let $a > b\mu$. In any counting the last vote counted could either be for A or for B . Thus we have the relation

$$N(a,b) = N(a-1,b) + N(a,b-1). \tag{1.3.3}$$

Equation (1.3.3) is a famous difference equation. A particular solution is

$$N(a,b) = \binom{a+b-1+j}{b-j} \qquad (j \leq b).$$

To check this we use Pascal identity and see that

$$\binom{a-1+b-1+j}{b-j} + \binom{a+b-2+j}{b-1-j} = \binom{a+b+j-1}{b-j} .$$

The general solution of (1.3.3) is

$$N(a,b) = \sum_{j=o}^{b} C_j \binom{a+b-1-j}{b-j}$$

where C_j's are constants to be determined by the boundary conditions.

If $a \geq 0$, then $N(a,0) = 1$ and thus $C_o=1$. If $a = [b\mu]$, then a moment's reflection shows that

$$N([b\mu],b) = 0 .$$

Thus the coefficients C_j $(j=1, \ldots, b)$ may be determined recursively from the formula

$$\sum_{j=o}^{b} C_j \binom{[b\mu]+b-1-j}{b-j} = 0 \qquad (b=1,2,\ldots)$$

where $C_o=1$. Since the C_j's are determined recursively they are unique. Thus we have

$$P(a,b) = \sum_{j=o}^{b} \frac{C_j\binom{a+b-1-j}{b-j}}{\binom{a+b}{a}}$$

i.e.

$$P(a,b) = \frac{a}{a+b} \sum_{j=o}^{b} C_j \frac{\binom{b}{j}}{\binom{a+b-1}{j}}$$

which completes the proof.

In particular, if μ is an integer, then we will show that $C_j = -\mu$ $(j=1,2,\ldots)$ and thus

$$P(a,b) = \frac{a-\mu b}{a+b}$$

which is a well known result. For $b=1$, we have

$$\sum_{j=o}^{1} C_j \binom{\mu-j}{1-j} = 0$$

which yields

$$C_1 = -\mu .$$

Assume then that $C_j = -\mu$ for $1 \leq j \leq k$. Then we have

$$\sum_{j=o}^{k+1} c_j \binom{(k+1)\mu+(k+1)-1-j}{k+1-j} = 0$$

i.e.

$$\sum_{j=1}^{k} (-\mu) \binom{(k+1)\mu+(k+1)-j-1}{k+1-j} + c_o \binom{(k+1)\mu+(k+1)-1}{k+1}$$

$$= -c_{k+1}\binom{(k+1)\mu+(k+1)-1-(k+1)}{0}$$

giving

$$c_{k+1} = \mu \sum_{j=1}^{k} \binom{(k+1)\mu+k-j}{(k+1)-j} - \binom{(k+1)\mu+k}{k+1} .$$

Now using the well known identity [cf. for example 5, p.62]

$$\sum_{j=o}^{k} \binom{a+k-j-1}{k-j}\binom{b+j-1}{j} = \binom{a+b+k-1}{k}$$

we have

$$\sum_{j=1}^{k} \binom{(k+1)\mu+k-j}{(k+1)-j} = \binom{(k+1)\mu+k+1}{k+1} - \binom{(k+1)\mu+k}{k+1} - 1 .$$

Thus

$$c_{k+1} = \mu \left\{\binom{(k+1)\mu+k+1}{k+1} - \binom{(k+1)\mu+k}{k+1} -1\right\} - \binom{(k+1)\mu+k}{k+1}$$

and now using the Pascal identity and noting that

$$\mu \binom{(k+1)\mu+k}{k} = \binom{(k+1)\mu+k}{k+1}$$

we have

$$C_{k+1} = -\mu$$

which proves our assertion.

In the general case we have

$$C_1 = -[\mu],\ C_2 = -\frac{[2\mu]}{2}(1-2[\mu]+[2\mu]),\ \text{etc.}$$

We remark in passing that we will give yet another characterization and solution of the ballot problem in the next chapter. Presently we wish to use the result of this theorem to study the fluctuation of the frequency of successes in a sequence of Bernoulli trials.

Consider a sequence of Bernoulli trials with probability p of success and denote by ν_n the number of successes in the first n trials. We wish to find out the distribution function of the random variable

$$\sup_{1 \leq n < \infty} \frac{\nu_n}{n} .$$

Theorem 1.3.2:

If $\mu < \frac{1-p}{p}$ then we have

$$(1.3.4) \qquad P\left\{ \frac{\nu_n}{n} < \frac{1}{\mu+1} \quad \text{for } n=1, 2, \dots \right\} = (1-p) \sum_{j=0}^{\infty} C_j\, p^j$$

where $C_0=1$ and the constants $C_j (j=1,2,\dots)$ are given by the formula (1.3.2).

Proof:

Let

$$\chi_n = \begin{cases} 1 & \text{if the } n^{th} \text{ trial is a success} \\ 0 & \text{if the } n^{th} \text{ trial is a failure.} \end{cases}$$

Then we note that

$$\nu_n = \sum_{j=1}^{n} \chi_j \quad .$$

Now consider the urn problem of section 1.2. Let

$$\chi_n(a,b) = \begin{cases} 1 & \text{if the } n^{th} \text{ number drawn is } \mu+1 \\ 0 & \text{if the } n^{th} \text{ number drawn is } 0 \ . \end{cases}$$

Then for $1 \leqq n_1 < n_2 < \ldots < n_k$ the joint distribution of $\chi_{n_1}(a,b),\ \chi_{n_2}(a,b),\ \ldots,\ \chi_{n_k}(a,b)$ [cf. 5, p. 109] approaches the joint distribution of the random variables $\chi_{n_1}, \chi_{n_2}, \ldots, \chi_{n_k}$ as $a \to \infty$, $b \to \infty$ in such a way that $\frac{b}{a+b} \to p$. Thus for every x and every finite positive integer N we have

$$P\left\{ \max_{1 \leqq n \leqq N} \frac{1}{n} \sum_{j=1}^{n} \chi_j(a,b) < x \right\} \to P\left\{ \max_{1 \leqq n \leqq N} \frac{1}{n} \sum_{j=1}^{n} \chi_j < x \right\}$$

if $a \to \infty$, $b \to \infty$ and $\frac{b}{a+b} \to p$. This relation remains valid if we let $N = \infty$.

Now we let $a \to \infty$, $b \to \infty$ in such a way that $\frac{b}{a+b} \to p$ in (1.3.1), we have

$$\lim_{\substack{a\to\infty \\ b\to\infty \\ b/(a+b)\to p}} P(a,b) = \lim \frac{a}{a+b} \sum_{j=o}^{b} C_j \frac{\binom{b}{j}}{\binom{a+b-1}{j}} = (1-p) \sum_{j=o}^{\infty} C_j p^j$$

But (1.3.1) is valid if and only if $a > b\mu$ i.e. if and only if

$$\mu < \frac{a/(a+b)}{b/(a+b)} = \frac{1-p}{p} .$$

Thus, if $\mu < \frac{1-p}{p}$ then we have

$$P\left\{ \frac{\nu_n}{n} < \frac{1}{1+\mu} \quad \text{for } n=1,2,\ldots \right\} = (1-p) \sum_{j=o}^{\infty} C_j p^j .$$

We do not propose to give any further applications of Theorems 1.2.1 and 1.2.2. The interested reader is referred to [3] for some applications of Theorems 1.2.1 and 1.3.1 in queueing theory.

§ 1.4 Related results in fluctuation theory.

In section 1.2 we reviewed some combinatorial theorems established by Takács. In this section we wish to review some recent results in fluctuation theory due to Graham [6] and Dwass [7]. These results, although of an elementary nature, will prove to be far more general than those of section 1.2. In fact, we will show that the results of section 1.2 follow directly from the theorems of this section.

Let us introduce the following notation: Let $(x_1, x_2, \ldots, x_n)$ denote a sequence of real numbers and $s_k = \sum_{j=1}^{k} x_j$ be the k^{th} partial sum. Let

$$x^+ = \begin{cases} x & \text{if} \quad x \geq 0 \\ 0 & \text{if} \quad x \leq 0 \end{cases}$$

$$x^- = \begin{cases} x & \text{if} \quad x \leq 0 \\ 0 & \text{if} \quad x \geq 0 \end{cases} .$$

Denote by $m(x_1, x_2, \ldots, x_k)$ the <u>r^{th} largest term</u> of $(x_1, x_2, \ldots, x_n)$ and let $m(x_1, x_2, \ldots, x_n) = 0$ if $r > n$.

The properties A, B, C, below follow immediately from the definitions and will be used often in the sequel.

A. If $x < 0$, $y < 0$ (and even if $x \neq y$), then $x^+ = 0 = y^+$; and if $x = y$ then $x^+ = y^+$.

B. $m(x_1, x_2, \ldots, x_n) \geq m(x_1, x_2, \ldots, x_{n-1})$ and $m(x_1, x_2, \ldots, x_n)^+ \geq m(x_1, x_2, \ldots, x_{n-1})^+$.

C. If s_p is the ν^{th} largest term $(\nu < r)$ and s_q the r^{th} largest term of $(s_1, s_2, \ldots, s_n)$ then $s_p \geq s_q$.

We are now in a position to prove the following

Lemma 1.4.1:

If $y \geq 0$ then

$$m(x_1,x_2,\ldots,x_n,y)^+ - m(x_1,x_2,\ldots,x_n)^+ \tag{1.4.1}$$

$$= m(x_1,x_2,\ldots,x_n,y) - m(x_1,x_2,\ldots,x_n,0) .$$

Proof:

There are three different cases to be considered.

(i) Let $m(x_1, x_2, \ldots, x_n, y) \leqq 0$.

In this case, by definition

$$m(x_1, x_2, \ldots, x_n, y)^+ = 0$$

and $m(x_1, x_2, \ldots, x_n) \leqq m(x_1, x_2, \ldots, x_n, y)$ by B . Thus $m(x_1, \ldots, x_n) \leqq 0$ and $m(x_1, \ldots, x_n)^+ = 0$.

Now $y \geqq 0$ and $m(x_1, x_2, \ldots, x_n, y) \leq 0$ together imply that y is not the rth largest term but is the vth $(v < r)$ largest term. It follows that if we replace y by an arbitrary non-negative real number ϵ , the rth largest term in $(x_1, x_2, \ldots, x_n, \epsilon)$ still remains the same and is $\leqq 0$. In particular, for $\epsilon = 0$ we have

$$m(x_1, x_2, \ldots, x_n, 0) = m(x_1, x_2, \ldots, x_n, y) .$$

Thus (1.4.1) holds.

(ii) Let $m(x_1, x_2, \ldots, x_n, y) > 0$ and $m(x_1, x_2, \ldots, x_n) > 0$. Then, by definition

$$m(x_1, x_2, \ldots, x_n, y)^+ = m(x_1, x_2, \ldots, x_n, y)$$

and

$$m(x_1, x_2, \ldots, x_n)^+ = m(x_1, x_2, \ldots, x_n) .$$

Also, $m(x_1, x_2, \ldots, x_n) > 0$ implies that the r^{th} largest term in $(x_1, x_2, \ldots, x_n, 0)$ is the same as that in $(x_1, x_2, \ldots, x_n)$. Thus

$$m(x_1,x_2,\ldots,x_n,0) = m(x_1,x_2,\ldots,x_n) = m(x_1,x_2,\ldots,x_n)^+ .$$

Thus (1.4.1) again holds.

(iii) Finally, let $m(x_1, x_2, \ldots, x_n, y) > 0$ and $m(x_1, x_2, \ldots, x_n) \leq 0$.

In this case our assumptions imply that

$$m(x_1,x_2,\ldots,x_n,y)^+ = m(x_1,x_2,\ldots,x_n,y)$$

and

$$m(x_1,x_2,\ldots,x_n)^+ = 0 .$$

Also $y \geq 0$, $m(x_1, x_2, \ldots, x_n) \leq 0$ and $m(x_1, x_2, \ldots, x_n, y) > 0$ imply that the (r-1)th largest term in $(x_1, x_2, \ldots, x_n)$ is now the r^{th} largest term in $(x_1, x_2, \ldots, x_n, y)$ and that the (r-1)th largest term of $(x_1, x_2, \ldots, x_n)$ is > 0. It, therefore, follows that

$$m(x_1,x_2,\ldots,x_n,0)^+ = 0$$

and (1.4.1) is again satisfied.

This completes the proof of the lemma.

We are now in a position to prove a simple theorem from which the theorems of section 1.2 may be derived. Let $1 \leq r \leq n$ and m_k denote the r^{th} largest term in $(s_1, s_2, \ldots, s_k)$ where $s_k = \sum_{j=1}^{k} x_j$. Thus

$$m_k = m(s_1, s_2, \ldots, s_k) \quad .$$

Then we have the

Theorem 1.4.1:

$\sum\limits_{\sigma}(m_n^+ - m_{n-1}^+) = s_n^+$ where σ ranges over all cyclic permutations of $(x_1, x_2, \ldots, x_n)$.

Proof:

Let us first consider the case when $s_n < 0$. Then $s_n^+ = 0$ by definition. We shall show that $m_n^+ - m_{n-1}^+ = 0$ for all permutations of x_i . The proof of the theorem will then be complete for the case $s_n < 0$.

By B, we know that the summands $m_n^+ - m_{n-1}^+$ are non-negative. Thus we only have to show that if $m_n^+ > 0$, $m_{n-1}^+ > 0$ then $m_n^+ = m_{n-1}^+$ and $m_n^+ - m_{n-1}^+ = 0$. Otherwise $m_n^+ = 0 = m_{n-1}^+$.

Now, if s_n is the rth largest term in $(s_1, s_2, \ldots, s_n)$ then $m_n = s_n$ and it follows that

$$m_n^+ = s_n^+ = 0 \; .$$

Furthermore,

$$m_{n-1} \leq m_n = s_n \quad \text{(by B)}$$

implies that

$$m_{n-1}^+ = 0 \quad .$$

Next suppose that s_n is the $(r+k)$th $(k \geq 1)$ largest term in

$(s_1, s_2, \ldots, s_n)$. Then $m_n = m_{n-1}$ and it follows that $m_n^+ = m_{n-1}^+$ (by A).

Finally, if s_n is the v^{th} $(v < r)$ largest term in $(s_1, s_2, \ldots, s_n)$ then $m_n < s_n$ (by C) and it follows that $m_n^+ = 0$ and $m_{n-1}^+ = 0$ (by B).

Thus, for $s_n < 0$ we have shown that

$$\sum_\sigma (m_n^+ - m_{n-1}^+) = 0 = s_n^+ .$$

Next suppose that $s_n \geqq 0$. In this case $s_n = s_n^+$. Note that

$$m_n = m(x_1, x_1+x_2, \ldots, x_1+x_2+\ldots+x_n) \tag{1.4.2}$$

$$= x_1 + m(0, x_2, x_2+x_3, \ldots, x_2+x_3+\ldots+x_n) .$$

Thus

$$\sum_\sigma (m_n^+ - m_{n-1}^+)$$

$$= \sum_\sigma (m(x_1,x_1+x_2,\ldots,x_1+x_2+\ldots+x_n)^+ - m(x_1,x_1+x_2,\ldots,x_1+x_2+\ldots+x_{n-1})^+)$$

$$= \sum_\sigma (m(x_1,x_1+x_2,\ldots,x_1+x_2+\ldots+x_n) - m(x_1,x_1+x_2,\ldots,x_1+x_2+\ldots+x_{n-1},0))$$

(using (1.4.1))

$$= \sum_\sigma (m_n - m(0,x_1,x_1+x_2,\ldots,x_1+x_2+\ldots+x_{n-1}))$$

$$= \sum_{\sigma} (x_1 + m(0,x_2,x_2+x_3,\dots,x_2+x_3+\dots+x_n)$$

$$- m(0,x_1,x_1+x_2,\dots,x_1+x_2+\dots+x_{n-1})) \qquad \text{(by (1.4.2))}$$

$$= (x_1 + m(0,x_2,\dots,x_2+x_3+\dots+x_n) - m(0,x_1,\dots,x_1+x_2+\dots+x_{n-1}))$$

$$+ (x_2 + m(0,x_3,\dots,x_3+x_4+\dots+x_n+x_1) - m(0,x_2,\dots,x_2+\dots+x_n))$$

$$+ \quad . \quad . \quad . \quad . \quad .$$

$$+ (x_j + m(0,x_{j+1},\dots,x_{j+1}+\dots+x_n+x_1+\dots+x_{j-1})$$

$$-m(0,x_j,\dots,x_j+\dots+x_n+x_1+\dots+x_{j-2}))$$

$$+ \quad . \quad . \quad . \quad . \quad .$$

$$+ (x_n + m(0,x_1,\dots,x_1+\dots+x_{n-1}) - m(0,x_n,\dots,x_n+x_1+\dots+x_{n-2}))$$

$$= x_1 + \dots + x_n$$

$$= s_n$$

$$= s_n^+ \ .$$

This completes the proof.

We now prove a more general theorem. There are some misprints in Graham's paper and the theorem as stated below therefore differs somewhat from Theorem 2 in Graham's paper.

Let $(x_1, x_2, \dots, x_{t+u})$ be a sequence of real numbers and let

$$m_j(k) = m(x_{k+1}, x_{k+1}+x_{k+2}, \dots, x_{k+1}+\dots+x_{k+j})$$

for $0 \leqq k \leqq t$ and $1 \leqq j \leqq u$.

Theorem 1.4.2:

If $1 \leq n \leq u$, $1 \leq r \leq n$ and $\sum_{j=1}^{n} x_{k+j} \geq 0$ for $1 \leq k \leq t$

then

$$(1.4.3) \quad \sum_{k=1}^{t} (m_n(k)^+ - m_{n-1}(k)^+) = m_n(t) - m_n(0) + \sum_{k=1}^{t} x_k .$$

Proof:

The proof of this theorem is not very much different from that of Theorem 1.4.1. As before, we note that

$$(1.4.4) \quad m_j(k) = m(x_{k+1}, x_{k+1}+x_{k+2}, \ldots, x_{k+1}+\ldots+x_{k+j})$$

$$= x_{k+1} + m(0, x_{k+2}, \ldots, x_{k+2}+\ldots+x_{k+j}) \quad .$$

Thus

$$\sum_{k=1}^{t} (m_n(k)^+ - m_{n-1}(k)^+)$$

$$= \sum_{k=1}^{t} (m(x_{k+1}, x_{k+1}+x_{k+2}, \ldots, x_{k+1}+\ldots+x_{k+n})^+$$

$$- m(x_{k+1}, x_{k+1}+x_{k+2}, \ldots, x_{k+1}+\ldots+x_{k+n-1})^+)$$

$$= \sum_{k=1}^{t} (m(x_{k+1}, x_{k+1}+x_{k+2}, \ldots, x_{k+1}+\ldots+x_{k+n})$$

$$- m(x_{k+1}, x_{k+1}+x_{k+2}, \ldots, x_{k+1}+\ldots+x_{k+n-1}, 0)) \text{ (using (1.4.1))}$$

$$= \sum_{k=1}^{t} (m_n(k) - m(0, x_{k+1}, x_{k+1}+x_{k+2}, \ldots, x_{k+1}+\ldots+x_{k+n-1}))$$

$$= \sum_{k=1}^{t} (x_{k+1} + m(0, x_{k+2}, \ldots, x_{k+2}+\ldots+x_{k+n}) - m(0, x_{k+1}, \ldots, x_{k+1}+\ldots+x_{k+n-1}))$$

(by (1.4.4))

$$= \sum_{k=1}^{t} x_k + (x_{t+1} - x_1) + \sum_{k=1}^{t} (m(0, x_{k+2}, \ldots, x_{k+2}+\ldots+x_{k+n})$$

$$- m(0, x_{k+1}, \ldots, x_{k+1}+\ldots+x_{k+n-1}))$$

$$= \sum_{k=1}^{t} x_k + (x_{t+1} + m(0, x_{t+2}, \ldots, x_{t+2}+x_{t+3}+\ldots+x_{t+n}))$$

$$- (x_1 + m(0, x_2, \ldots, x_2+\ldots+x_n))$$

$$= \sum_{k=1}^{t} x_k + m(x_{t+1}, x_{t+1}+x_{t+2}, \ldots, x_{t+1}+\ldots+x_{t+n}) - m(x_1, x_1+x_2, \ldots, x_1+\ldots+x_n)$$

$$= \sum_{k=1}^{t} x_k + m_n(t) - m_n(0)$$

which proves the assertion.

Note that the assumption $\sum_{j=1}^{n} x_{k+j} \geq 0$, $1 \leq k \leq t$ has been used implicitly in the proof wherever we have used equation (1.4.1).

Corollary:

In equation (1.4.3) let $t = n$ and $x_{n+j} = x_j$ for $1 \leq j \leq n$. First we note that this assumption gives us all the n cyclic permutations of $(x_1, x_2, \ldots, x_n)$ on the left hand side of equation (1.4.3). For instance, $k = i$ gives $(x_{i+1}, \ldots, x_n, x_{1+n}, \ldots, x_{i+n})$ which is the permutation $(x_{i+1}, \ldots, x_n, x_1, \ldots, x_i)$ of $(x_1, \ldots, x_n)$. Thus we can write

$$\sum_{k=1}^{n} (m_n(k)^+ - m_{n-1}(k)^+) = \sum_{\sigma} (m_n^+ - m_{n-1}^+) \ .$$

Next we note that the assumption $\sum_{j=1}^{n} x_{k+j} \geq 0$ is the same as $\sum_{j=1}^{n} x_j \geq 0$, i.e. $s_n \geq 0$. Finally, we note that

$$\begin{aligned} m_n(n) &= m(x_{n+1}, \ldots, x_{n+1} + \ldots + x_{2n}) \\ &= m(x_1, \ldots, x_1 + \ldots + x_n) \\ &= m_n(0) \ . \end{aligned}$$

Hence equation (1.4.3) reduces to

$$\sum_{\sigma} (m_n^+ - m_{n-1}^+) = \sum_{j=1}^{n} x_j = s_n = s_n^+$$

which is what Theorem 1.4.1 asserts.

The next theorem is yet another generalisation of Theorem 1.4.1. The proof is exactly similar and we will omit it.

Theorem 1.4.3:

Let $(x_1, x_2, \ldots, x_n)$ be a sequence of real numbers and let $1 \leq m \leq n$. Suppose that the sum of any m consecutive x_j is non-negative where the x_j are considered cyclic, i.e. x_1 follows x_n, etc. Then for $m-1 \leq q \leq p \leq n$ we have

$$\sum_{\sigma} (m_p^+ - m_q^+) = (p-q)\ s_n^+ \quad . \tag{1.4.5}$$

We remark in passing that if we let $p = n$ and $q = n-1$ in Theorem 1.4.3 we obtain Theorem 1.4.1.

We now derive Theorems 1.2.1 and 1.2.2 from Theorem 1.4.1. Let us suppose that $(x_1, x_2, \ldots, x_n)$ is a sequence of integer-valued cyclic random variable such that

$$x_1 + x_2 + \ldots + x_n = 1 \quad .$$

Then Theorem 1.2.2 asserts that for any integer r, with $1 \leq r \leq n$ we have

$$P\{\Delta_n = r \,|\, x_1 + x_2 + \ldots + x_n = 1\} = \frac{1}{n}$$

where Δ_n is the number of positive partial sums amongst $s_1, s_2, \ldots, s_n$.

Now, under these conditions Theorem 1.4.1 asserts that $\sum_{\sigma}(m_n^+ - m_{n-1}^+) = s_n = 1$. We know that each summand $(m_n^+ - m_{n-1}^+)$ is a non-negative integer. It therefore follows that there must be <u>exactly one</u> cyclic permutation of $(x_1, x_2, \ldots, x_n)$ for which $(m_n^+ - m_{n-1}^+) > 0$. This inequality holds if and only if there are exactly r of the positive partial sums s_j which are $\geqq s_n = 1$. Thus there must be exactly r positive partial sums $1 \leqq r \leqq n$. This is what Theorem 1.2.2 asserts.

To derive Theorem 1.2.1 we interpret it as follows: If $(x_1, x_2, \ldots, x_n)$ is a sequence of non-negative integers with $x_1 + x_2 + \ldots + x_n = k \leq n$ then there are exactly $n-k$ cyclic permutations of $x_1, x_2, \ldots, x_n$ such that the j^{th} partial sum is less than j for $j = 1, 2, \ldots, n$.

Let us replace x_k by $(1 - x_{n+1-k})$ for $k = 1, 2, \ldots, n$. Then

we obtain the sequence of integers $(1 - x_n, 1 - x_{n-1}, \ldots, 1 - x_1)$ in which $1 - x_j \leqq 1$ for all j. Note that there is a one to one correspondence between the cyclic permutations of $(1 - x_n, 1 - x_{n-1}, \ldots, 1 - x_1)$ and of $(x_1, x_2, \ldots, x_n)$. Now $\sum_{j=1}^{n} (1 - x_j) = n - k > 0$ so that we can apply Theorem 1.4.1. Let $r = 1$, that is, let $m(1 - x_n, 1 - x_{n-1}, \ldots, 1 - x_1)$ be the largest of the terms in $(1 - x_n, \ldots, 1 - x_1)$. Then Theorem 1.4.1 asserts that

$$\sum_{\sigma} (m_n^+ - m_{n-1}^+) = n - k$$

where σ ranges over all the cyclic permutations of $(1 - x_n, \ldots, 1 - x_1)$ and hence of $(x_1, x_2, \ldots, x_n)$. We note that m_n^+ and m_{n-1}^+ are non-negative integers and $m_n^+ - m_{n-1}^+ \geqq 0$.

Now observe that the last term in any permutation is 0 or 1 or a negative integer. If the last term in any permutation is 0 or a negative integer then it follows that $m_n^+ = m_{n-1}^+$. If the last term is 1 then either s_n is the largest term or not. If s_n is the largest term, then $m_n^+ - m_{n-1}^+$ can at most be 1 (it may be zero) because at least $s_{n-1} = n - k - 1$. If s_n is not the largest term then $m_n^+ = m_{n-1}^+$. Hence we see that $m_n^+ - m_{n-1}^+$ can at most be 1. Thus there must be exactly $(n-k)$ cyclic permutations in which $m_n^+ - m_{n-1}^+$ is equal to 1. In these $(n-k)$ cyclic permutations $s_n = n - k$ is the largest partial sum and it follows that

$$s_j < n - k \qquad \text{for } j = 1, 2, \ldots, n-1$$

i.e.

$$(1 - x_n + 1 - x_{n-1} + \dots + 1 - x_{n-j+1}) < n-k$$

or

$$j - (x_{n-j+1} + \dots + x_n) < n-k$$

or

$$j - (\sum_{i=1}^{n} x_i - x_{n-j} - \dots - x_1) < n-k$$

or

$$j - k + (x_1 + \dots + x_{n-j}) < n-k$$

or

$$x_1 + x_2 + \dots + x_{n-j} < n-j \qquad \text{for} \quad j = 1, 2, \dots, n-1 .$$

Noting that $x_1 + \dots + x_n = k < n$ we have

$$x_1 + x_2 + \dots + x_j < j \qquad \text{for} \quad j = 1, 2, \dots, n .$$

Hence there are <u>exactly</u> $(n-k)$ cyclic permutations of $(x_1, x_2, \dots, x_n)$ such that

$$x_1 + x_2 + \dots + x_j < j \qquad \text{for} \quad j = 1, \dots, n .$$

Finally, we prove a theorem due to Meyer Dwass for cyclic sets of random variables. We shall call $X_1, X_2, \dots, X_n$ a cyclic set of random variables if $P(X_1 \leq t_1, \dots, X_n \leq t_n)$ is constant for all n cyclic permutations of the sequence $(t_1, t_2, \dots, t_n)$. Loosely speaking, the random variables are cyclic if their distribution law is invariant under cyclic permutations. Similarly, the set is called exchangeable (or symmetrically

dependent) if their distribution law is invariant under <u>all</u> permutations. It may be noted that exchangeable sets of random variables are cyclic, but the converse is not true. Let $X_1, X_2, \ldots, X_n$ be a cyclic set of random variables and let $m(X_1, X_2, \ldots, X_n)$ denote the maximum of $(X_1, X_2, \ldots, X_n)$. For simplicity let

$$S_k = X_1 + X_2 + \ldots + X_k$$

and

$$M = m(S_1, S_2, \ldots, S_n) .$$

Then we have the

Theorem 1.4.4:

$$E(M^-/S_n = s) = s^-/n \quad . \tag{1.4.6}$$

Proof:

We first remark that when $S_n = s \geqq 0$ either side of (1.4.6) vanishes.

Let us therefore assume that $s < 0$. We first give a numerical example to motivate the proof. Suppose that $(X_1, X_2, \ldots, X_6)$ is equally likely to be any of the six permutations of $(-\frac{3}{4}, -\frac{7}{3}, -1, 2, -\frac{1}{2}, 1, -\frac{1}{12})$. Table I lists all possible values of the relevant variables.

Table I

X_1	X_2	X_3	X_4	X_5	X_6	X_7	S_1	S_2	S_3	S_4	S_5	S_6	S_7	M^-
$-\frac{3}{4}$	$-\frac{7}{3}$	-1	2	$-\frac{1}{2}$	1	$-\frac{1}{12}$	$-\frac{3}{4}$	$-\frac{37}{12}$	$-\frac{49}{12}$	$-\frac{25}{12}$	$-\frac{31}{12}$	$-\frac{19}{12}$	$-\frac{5}{3}$	$-\frac{3}{4}$
$-\frac{7}{3}$	-1	2	$-\frac{1}{2}$	1	$-\frac{1}{12}$	$-\frac{3}{4}$	$-\frac{7}{3}$	$-\frac{10}{3}$	$-\frac{4}{3}$	$-\frac{11}{6}$	$-\frac{5}{6}$	$-\frac{11}{12}$	$-\frac{5}{3}$	$-\frac{5}{6}$
-1	2	$-\frac{1}{2}$	1	$-\frac{1}{12}$	$-\frac{3}{4}$	$-\frac{7}{3}$	-1	1	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{17}{12}$	$\frac{8}{12}$	$-\frac{5}{3}$	0
2	$-\frac{1}{2}$	1	$-\frac{1}{12}$	$-\frac{3}{4}$	$-\frac{7}{3}$	-1	2	$\frac{3}{2}$	$\frac{5}{2}$	$\frac{29}{12}$	$\frac{5}{3}$	$-\frac{2}{3}$	$-\frac{5}{3}$	0
$-\frac{1}{2}$	1	$-\frac{1}{12}$	$-\frac{3}{4}$	$-\frac{7}{3}$	-1	2	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{5}{12}$	$-\frac{1}{3}$	$-\frac{8}{3}$	$-\frac{11}{3}$	$-\frac{5}{3}$	0
1	$-\frac{1}{12}$	$-\frac{3}{4}$	$-\frac{7}{3}$	-1	2	$-\frac{1}{2}$	1	$\frac{11}{12}$	$\frac{1}{6}$	$-\frac{13}{6}$	$-\frac{19}{6}$	$-\frac{7}{6}$	$-\frac{5}{3}$	0
$-\frac{1}{12}$	$-\frac{3}{4}$	$-\frac{7}{3}$	-1	2	$-\frac{1}{2}$	1	$-\frac{1}{12}$	$-\frac{5}{6}$	$-\frac{19}{6}$	$-\frac{25}{6}$	$-\frac{13}{6}$	$-\frac{8}{3}$	$-\frac{5}{3}$	$-\frac{1}{12}$

$$k_1 = 1,\ k_2 = 5,\ k_3 = 1$$

First observe that

$$7E(M^-) = [(-\tfrac{3}{4}) + (-\tfrac{5}{6}) + 0 + 0 + 0 + 0 + (-\tfrac{1}{12})] = -\tfrac{5}{3} = S_7$$

as asserted. Now note that in the first permutation M^- is achieved in the first (k_1) position. The value of M^- in the following permutations up to but excluding the second (k_1+1) is zero. In the second (k_1+1) permutation M^- is achieved in the fifth (k_2) position, and the value of M^- in the following permutations upto but excluding the seventh (k_1+k_2+1) is 0. In the seventh (k_1+k_2+1) permutation, M^- is achieved in the first $(k_3=1)$ position. Since $k_1 + k_2 + k_3 = 7 = n$ we stop the process. We thus have

$$\begin{aligned} 7E(M^-) &= -\frac{3}{4} - \frac{5}{6} - \frac{1}{12} \\ &= (-\frac{3}{4}) + (-\frac{7}{3} - 1 + 2 - \frac{1}{2} + 1) - \frac{1}{12} \\ &= S_7 \end{aligned}$$

Note that each x_i occurs exactly once in some negative M^- .

Now we prove the theorem formally. We first note that it is sufficient to prove the theorem assuming all the mass to be concentrated on the n cyclic permutations of a given set of numbers, $x_1, x_2, \ldots, x_n$ (as in the above example) . Denote the cyclically permuted sequence $x_k, \ldots, x_{k-1}$ by $T(k)$ and the maximum of partial sums in $T(k)$ by $m(k)$. The proof consists in showing that each x_i occurs exactly once in some negative $m(k)$.

We first prove that since $s_n < 0$, $m(k)$ must be negative for some k . To see this let us assume that $m(k) \geqq 0$ for some k . If no such k exists then there is nothing to prove. Thus, setting $x_{n+\nu} = x_n$ if $k + j > n$,

$$x_k + \ldots + x_{k+j} \geqq 0$$

for some positive integer $j < n-1$ (note that for $j = n-1$, this inequality does not hold, for then $s_n < 0$). It therefore follows that

$$x_1 + x_2 + \ldots + x_{k-1} + x_{k+j+1} + \ldots + x_n < 0$$

with

$$|x_k + \ldots + x_{k+j}| < |x_1 + \ldots + x_{k-1} + x_{k+j+1} + \ldots + x_n| .$$

Now consider the permutation $T(k + j + 1)$,

$$x_{k+j+1}, \ldots, x_n, x_1, \ldots, x_{k-1}, x_k, \ldots, x_{k+j} .$$

In this sequence, the subsequence $x_k, \ldots, x_{k+j}$ has the maximum non-negative sum and also $s_n < 0$. A moment's reflection shows that $m(k + j + 1) < 0$. This proves our assertion.

Thus there is no loss of generality in assuming that $m(1) < 0$. This implies that $s_k < 0$ for $k = 1, 2, \ldots n$. The rest of the proof consists of the following steps:

a) For any k, the partial sums of $T(k+1)$ are

$$x_{k+1}, x_{k+1} + x_{k+2}, \ldots, x_{k+1} + \ldots + x_k$$

or

(1.4.7) $$s_{k+1}-s_k, \ldots, s_n-s_k, s_n-s_k+s_1, \ldots, s_n-s_k+s_k .$$

Since s_n and $s_k(k = 1, \ldots, n)$ are both negative, it follows that $s_n - s_k > s_n - s_k + s_i$ $(i = 1, \ldots, k)$. This implies that $m(k+1)$ is achieved for the last time by one of the $(n-k)$ terms, $s_{k+1} - s_k, \ldots, s_n - s_k$.

b) Let k_1 be the position where $m(1)$ is last achieved. Then $s_{k_1} \geqq s_k$ for $k < k_1$ and $s_{k_1} > s_k$ for $k > k_1$. Now $s_{k_1} > s_k$ for $k > k_1$ implies that $s_{k_1} - s_{k_1+1} > 0, \ldots, s_{k_1} - s_n > 0$ and thus by (a) above $m(k_1+1) < 0$. Further, $s_{k_1} \geq s_k$ for $k < k_1$ implies that $s_{k_1} - s_{k_1-1} \geqq 0, \ldots, s_{k_1} - s_2 \geqq 0$ and thus $m(k) \geqq 0$ for $2 \leqq k \leqq k_1$.

For $k = 1$, by assumption $m(1) < 0$.

c) Let k_2 be the position in $T(k_1+1)$ where $m(k_1+1)$ is last achieved. From (a), $k_2 \leqq n-k_1$. Applying the same arguments as in (b) , we have that

$$m(k_1+k_2+1) < 0 , \quad \text{and} \quad m(k) \geqq 0 \quad \text{if} \quad k_1+1 < k \leqq k_1+k_2 .$$

d) The above procedure is continued for a finite number of steps until $k_1 + k_2 + \ldots + k_r = n$.

e) Finally we have

$$m(k_1) = s_{k_1}$$

$$m(k_2) = s_{k_1+k_2} - s_{k_1}$$

$$\vdots$$

$$m(k_r) = s_{k_1+k_2+\ldots+k_r} - s_{k_1+k_2+\ldots+k_{r-1}} ,$$

and so

$$nE(M^-) = m(k_1) + \ldots + m(k_r) = s_n = s_n^- ,$$

which completes the proof.

Next we study an interesting special case of (1.4.6) when the X_i assign all their mass to $-1, 0, 1, 2, \ldots$. Theorem 1.2.1 will follow as a special case of this. Under the condition that $s_n = u < 0$,

and $M < 0$, we must have that $M = -1$. In this special case (1.4.6) states that

$$P(M < 0 | S_n = u) = -\frac{u}{n} \quad . \tag{1.4.8}$$

Let $Y_1, Y_2, \ldots, Y_n$ be cyclic random variables and let each Y_i assign all its mass to $0, 1, 2, \ldots$. Then $Y_1-1, \ldots, Y_n-1$ are also cyclic random variables and assign all their mass to $-1, 0, 1, 2, \ldots$. If $Y_1 + Y_2 + \ldots + Y_n = r \leqq n$, then $\sum_{i=1}^{n} (Y_i-1) = r-n \leqq 0$. Let M be the maximum term of $(Y_1-1, Y_1-1 + Y_2-1, \ldots, Y_1-1 + \ldots + Y_n-1)$ and let $M < 0$. Then applying (1.4.8) we get

$$P(M < 0 | Y_1-1 + Y_2-1 + \ldots + Y_n-1 = r-n) = \frac{n-r}{n}.$$

Now $M < 0$ implies that all partial sums of the sequence $(Y_1-1, Y_2-1, \ldots, Y_n-1)$ are negative, i.e.

$$Y_1-1 + \ldots + Y_k-1 < 0 \qquad \text{for} \quad k = 1, \ldots, n ,$$

$$Y_1 + \ldots + Y_k < k \qquad \text{for} \quad k = 1, \ldots, n .$$

Hence we have

$$P(Y_1 + \ldots + Y_k < k, k = 1, \ldots, n | Y_1 + \ldots + Y_n = r) = 1-\frac{r}{n} \tag{1.4.9}$$

which is (1.2.1) .

This concludes our discussion of some combinatorial problems arising in queueing theory and fluctuation theory. Some of the results

obtained in this chapter, particularly the ballot theorem, will be rederived in the next chapter using an entirely different approach. However, no reference will be made to the theorems of this chapter in the rest of this thesis.

CHAPTER II

AN ANALOGUE OF THE MULTINOMIAL THEOREM.

§ 2.1 Introduction

In Chapter I we saw that the urn problem of Takács can be reformulated as a generalised ballot problem. We also showed that Takács' results of section 1.2 can be derived from the elegant theorems of Graham and Dwass. In this chapter we will mainly concern ourselves with a theorem proved by Narayana in [8] which suggests a unified approach to ballot theorems as well as several other problems concerning lattice paths.

We motivate our analogue of the multinomial theorem by an informal discussion of multinomial coefficients in section 2.2. More precisely, we define the multinomial coefficients in terms of a recursive relation. In the rest of this section we discuss some basic results pertaining to the analogue while in the next section we interpret ballot theorems in the light of this discussion. Sections 2.4 and 2.5 extend the results in [8] to yield a refinement of ballot problems in a different direction than [3,4]. In section 2.6 we obtain solutions of some other related combinatorial problems as a by-product of our approach.

Finally, we remark that we have carefully looked into the possibility of deriving the results of this chapter from those of Chapter I (or vice versa) but we have not been able to do so.

§ 2.2 An analogue of the multinomial theorem and some preliminaries

Let n , x_i $(i = 1, 2, \ldots, k)$ be non-negative integers. Then the expression $(a_1 + a_2 + \ldots + a_k)^n$ can be expressed by the multinomial theorem in the form

$$(2.2.1) \qquad \sum_{x_1=0}^{n} \sum_{x_2=0}^{n} \cdots \sum_{x_k=0}^{n} \binom{n}{x_1, x_2, \ldots, x_k} \prod_{i=1}^{k} a_i^{x_i} .$$

The properties of the multinomial coefficients $\binom{n}{x_1, x_2, \ldots, x_k}$ are well known and we do not intend to discuss them here. However, we wish to emphasize that $\binom{n}{x_1, x_2, \ldots, x_k} = 0$ if $\sum_{i=1}^{k} x_i \neq n$ and $\binom{0}{0, 0, \ldots, 0} = 1$.

Now consider the function defined recursively, for $n \geqq 1$, as follows:

$$(0; 0, 0, \ldots, 0)^* = 1$$

$$(2.2.2) \qquad (n; x_1, \ldots, x_k)^* = \begin{cases} 0 & \text{if } \Sigma x_i \neq n , \\ \sum_{j=1}^{k} (n-1; x_1, \ldots, x_{j-1}, x_j - 1, x_{j+1}, \ldots, x_k)^* , & \text{otherwise.} \end{cases}$$

Then it is easily verified that

$$(2.2.3) \qquad (n; x_1, x_2, \ldots, x_k)^* = \binom{n}{x_1, x_2, \ldots, x_k} ,$$

so that (2.2.2) can be considered a recursive definition of the multinomial coefficients.

It is then legitimate to ask: Do there exist any analogous functions, non-vanishing for $\sum_i x_i \neq n$, defined recursively which have interesting combinatorial interpretations? As one example, we define below such a function which will prove to have interesting applications.

In what follows n, x_i, y_i $(i = 1, 2, \ldots, k)$ are non-negative integers. For $n \geqq 1$, consider the function $(n; x_1, x_2, \ldots, x_k)$ defined recursively as follows:

$$(0; 0, 0, \ldots, 0) = 1$$

$$(n; x_1, x_2, \ldots, x_k) = \begin{cases} 0 & \text{if } \sum_{i=1}^{n} x_i > n , \\ \sum_{y_1=0}^{x_1} \cdots \sum_{y_k=0}^{x_k} (n-1; y_1, y_2, \ldots, y_k) , & \text{otherwise} . \end{cases} \tag{2.2.4}$$

Then $(n; x_1, x_2, \ldots, x_k)$, defined in (2.2.4), is explicitly given by the following

Theorem 2.2.1:

When $\sum_i x_i > n$,

$$(n; x_1, x_2, \ldots, x_k) = 0$$

while for $\sum_i x_i \leqq n$,

$$(n; x_1, x_2, \ldots, x_k) = \prod_{i=1}^{k} \binom{n + x_i}{x_i} \left[1 - \frac{\sum_i x_i}{n+1} \right] . \tag{2.2.5}$$

Proof:

The first statement (when $\sum_i x_i > n$) is trivially true by <u>definition</u> (2.2.4), and (2.2.5) constitutes the theorem. A detailed proof of this theorem is contained in [8] where some properties of $(n; x_1, x_2, \ldots, x_k)$ are also discussed. We give below an equivalent proof using the principle of induction.

For $n = 0$, either side of (2.2.5) equals 1 and for $n = 1$, a simple calculation shows that either side of (2.2.5) again equals 1. Thus (2.2.5) holds for $n = 0, 1$. Suppose that (2.2.5) holds for $n = m$ $(m \geqq 1)$ and $\sum_i x_i \leqq m$, we will prove that it holds for $n = m+1$ and $\sum_i x_i \leqq m+1$.

By (2.2.4)

$$(m+1; x_1, x_2, \ldots, x_k) = \sum_{y_1=o}^{x_1} \cdots \sum_{y_k=o}^{x_k} (m; y_1, \ldots, y_k) \text{ , for } \sum_i x_i \leq (m+1) .$$

Since $\sum_i x_i \leq m+1$, we have $\sum_i y_i < m+1$ unless $y_1 = x_1, y_2 = x_2, \ldots, y_k = x_k$ and $x_1 + x_2 + \ldots + x_k = m+1$. This is a single term and, by (2.2.4) has value 0. Hence we can suppose that $\sum_i y_i \leqq m$, in which case we can apply (2.2.5) to yield

$$(m+1; x_1, x_2, \ldots, x_k) = \sum_{y_1=o}^{x_1} \cdots \sum_{y_k=o}^{x_k} \prod_{i=1}^{k} \binom{m + y_i}{y_i} \left[1 - \frac{\sum_i y_i}{m+1}\right] .$$

Taking the factor $\binom{m + y_1}{y_1}$ inside the square brackets and breaking the right hand side into two finite sums we obtain

$$(m+1;x_1,x_2,\ldots,x_k) = \sum_{y_2=o}^{x_2} \cdots \sum_{y_k=o}^{x_k} \prod_{i=2}^{k} \binom{m+y_i}{y_i} \left[\sum_{y_1=o}^{x_1} \left(1 - \frac{\sum_i y_i}{m+1}\right) \binom{m+y_1}{y_1} \right]$$

$$- \sum_{y_2=o}^{x_2} \cdots \sum_{y_k=o}^{x_k} \prod_{i=2}^{k} \binom{m+y_i}{y_i} \left[\sum_{y_1=o}^{x_1} \frac{y_1}{m+1} \binom{m+y_1}{y_1} \right]$$

$$= \sum_{y_2=o}^{x_2} \cdots \sum_{y_k=o}^{x_k} \prod_{i=2}^{k} \binom{m+y_i}{y_i} \left[\left(1 - \frac{\sum_i y_i}{m+1}\right) \binom{m+x_1+1}{x_1} - \binom{m+x_1+1}{x_1-1} \right] .$$

We have used the well-known identity

$$\sum_{v=o}^{r} \binom{v+k-1}{k-1} = \binom{r+k}{k}$$

[cf. 5, pp.62] to obtain the last step. Since this identity will be used quite frequently in the sequel we will refer to it as I . Thus,

$$(m+1;x_1,x_2,\ldots,x_k) = \binom{m+x_1+1}{x_1} \sum_{y_2=o}^{x_2} \cdots \sum_{y_k=o}^{x_k} \prod_{i=2}^{k} \binom{m+y_i}{y_i} \left[1 - \frac{\sum_{i=2}^{k} y_i}{m+1} - \frac{x_1}{m+2} \right] .$$

Repeating for $i = 2, 3, \ldots, k$ we obtain, with $\sum_i x_i \leq m+1$,

$$(m+1;x_1,x_2,\ldots,x_k) = \prod_{i=1}^{k} \binom{m+x_i+1}{x_i} \left[1 - \frac{\sum_i x_i}{m+2} \right] .$$

This completes the proof.

It is now clear that $(n; x_1, x_2, \ldots, x_k)$ does not vanish for $\sum_i x_i \leq n$.

Proceeding similarly to (2.2.5), we can establish, for $\Sigma x_i \leq n$,

$$(2.2.6) \quad \sum_{y_1=o}^{x_1} (n;y_1,x_2,\ldots,x_k) = \binom{n+x_1+1}{x_1} \prod_{i=2}^{k} \binom{n+x_i}{x_i} \left[1 - \frac{x_1}{n+2} - \frac{\sum_{i=2}^{k} x_i}{n+1}\right]$$

We wish to emphasize that summations of the type occurring in the left-hand side of (2.2.6) are basic to our later development. Such summations will be repeatedly used in the sequel. We shall therefore find it convenient to utilize a simple notation for them. Generally, let S^* be the subset $[i_1, i_2, \ldots, i_r]$ of the integers $[1, 2, \ldots, k]$ where $1 \leq i_1 < i_2 < \ldots < i_r \leq k$, and let S be the complementary set.

For $\sum_i x_i \leq n$, we denote the sum

$$\sum_{y_{i_1}=o}^{x_1} \cdots \sum_{y_{i_r}=o}^{x_r} (n; x_1,\ldots,y_{i_1},\ldots,y_{i_r},\ldots,x_k)$$

simply by $(n; x_1, \ldots, x^*_{i_1}, \ldots, x^*_{i_r}, \ldots, x_k)$ where the asterisks are placed only on those x's whose subscripts belong to S^* . If $\sum_{j=1}^{r} x_{i_j} = s^*$ and $\sum_{j\epsilon S} x_j = s$, we can establish similarly to (2.2.5) the general identity.

$$(2.27) \quad (n;x_1,\ldots,x^*_{i_1},\ldots,x^*_{i_r},\ldots,x_k) = \prod_{j=1}^{r} \binom{n+x_{i_j}+1}{x_{i_j}} \prod_{j\epsilon S} \binom{n+x_j}{x_j} \left[1 - \frac{s^*}{n+2} - \frac{s}{n+1}\right]$$

Following this notation we can write

$$(n-1;x_1^*,x_2^*,\ldots,x_k^*) = (n;x_1,\ldots,x_k) \quad \text{for} \quad \Sigma x_i \leq n-1$$

$$(n;x_1^*,x_2,\ldots,x_k) = \sum_{y_1=o}^{x_1} (n;y_1,x_2,\ldots,x_k) \quad .$$

Finally, we remark that we will later extend our definitions of the left-hand side of (2.2.7) to cover cases in which $\sum_i x_i = n+1$.

§ 2.3 Some interpretations

In this section we prove a preliminary lemma which establishes the connection between $(n;x_1,\ldots,x_k)$ and a certain set of vectors to be defined below. Before doing so we wish to characterize a lattice path from $(0, 0)$ to (m, n) $(m > nk)$ as a vector of non-negative integers.

It is well known [cf. for example 5, p.66] that the ballot theorem is equivalent to proving that the number of lattice paths from $(0, 0)$ to (m, n) $(m > nk)$, which do not touch the line $x = ky$ except at $(0, 0)$ is $\frac{m - nk}{m+n}\binom{m+n}{n}$. For simplicity, in what follows, when we refer to "path" we always mean a lattice path from $(0, 0)$ to (m, n) which does not touch the line $x = ky$ except at $(0, 0)$.

Let us first confine ourselves to points $(nk+1, n)$, $n \geqq 2$. We note that every path from $(0, 0)$ to $(nk+1, n)$ must have passed through $(nk+1, n-1)$ (since for $n \geqq 2$ it is impossible for a path to reach (nk,n) violating our definition of path); in particular, the number

of paths to $(nk+1, n)$ and $(nk+1, n-1)$ is the same.

For notational simplicity, we study paths to points $(nk+1, n-1)$ for $n \geqq 2$ or equivalently to points $P_n = (nk+k+1, n)$ for $n \geqq 1$. With any path from $(0, 0)$ to P_n we associate the "path vector" $(a_1, a_2, \ldots, a_n)$ where a_i represents the distance measured parallel to the x-axis, of the path from the point $(nk+k+1, n-i)$. We remark that various other characterizations of "path vector" are possible. For instance, we could take a_i to represent the distance measured parallel to the y-axis, of the path from the point $(i, 0)$. We will however use the first characterization because it is sufficient for our purpose.

Consider now the set $A_{n,k}$ of vectors of non-negative integers $A_n = (a_1, a_2, \ldots, a_n)$ whose elements satisfy

$$(2.3.1) \qquad \begin{cases} 0 \leqq a_1 \leqq \cdots \leqq a_n \\ 0 \leqq a_i \leqq ki & i = 1, 2, \ldots, n \end{cases}$$

We assert that to every vector A_n in $A_{n,k}$ corresponds a path from $(0, 0)$ to P_n and conversely. To see this, consider the "triangle" $\triangle$ bounded by

$$0 \leqq ky \leqq x \leqq nk + k + 1$$

and let $P(n)$ denote a path from $O(0, 0)$ to $P_n(nk + k + 1, n)$ of the form

$$\left\{O\ R_o\ R_o'\ R_1\ R_1' \ldots\ldots R_{n-1}\ R_{n-1}'\ P_n\right\}$$

where

$$R_{n-j} = (nk + k + 1 - a_j,\ n-j), \quad j = 1, \ldots, n ,$$

and

$$R'_{n-j} = R_{n-j} + (0,\ 1) , \qquad j = 1, 2, \ldots, n \ .$$

A moment's reflection now shows that $P(n)$ lies inside $\triangle$ and to every path of the type $P(n)$ corresponds a unique path vector $(a_1, a_2, \ldots, a_n)$ satisfying (2.3.1) and conversely, to every path vector in $A_{n,k}$ corresponds a path from $(0,0)$ to P_n .

We remark that a binary relation of "domination" D may be defined on the set of all lattice paths to P_n . If a, b are two paths to P_n then we say that a dominates b (written a D b) if no part of b lies between a and the line $x = ky$. D is a partial order that can be extended to the set $A_{n,k}$ and the one to one correspondence between vectors in $A_{n,k}$ and paths to P_n shows that the set of all lattice paths to P_n and $A_{n,k}$ are isomorphic partially ordered sets. Due to this isomorphism we are able to consider only vectors wherever convenient. This approach was initiated by Narayana in [9] and was used by him and Mohanti in [10] to yield a unified approach to many problems.

Next we introduce the subset $[n; x_1, \ldots, x_k]$ of $A_{n,k}$ such that every vector in $[n; x_1, \ldots, x_k]$ has exactly x_i of its positive elements congruent to $i (\mathrm{mod}\ k)$, $i = 1, 2, \ldots, k$, respectively.

Illustration: For $n = 3,\ k = 2$, the set $A_{3,2}$ consists of the following vectors:

000, 001, 002, 003, 004, 005, 006, 011, 012, 013, 014, 015,

016, 022, 023, 024, 025, 026, 033, 034, 035, 036, 044, 045,

046, 111, 112, 113, 114, 115, 116, 122, 123, 124, 125, 126,

133, 134, 135, 136, 144, 145, 146, 222, 223, 224, 225, 226,

233, 234, 235, 236, 244, 245, 246.

The vector $(122) \in A_{3,2}$ belongs to the subset $[3;\ 1,\ 2]$. It has three positive elements, and exactly one of these elements is $\equiv 1 \pmod 2$ while exactly two are $\equiv 2 \pmod 2$. The remaining elements of $[3;1,2]$ are 124, 126, 144, 146, 223, 225, 234, 236, 245.

We are now in a position to prove the

Lemma 2.3.1:

The number of vectors in $[n;x_1,\ \ldots,\ x_k]$ is $(n;\ x_1,\ \ldots,\ x_k)$ where $(n;\ x_1,\ \ldots,\ x_k)$ is given by equation (2.2.5) for $n \geqq 1$.

Proof:

A detailed, though slightly different, proof of this lemma is contained in [8]. We give below a shorter proof followed by an illustration.

For $n = 1$, the lemma is easily verified. For $n > 1$ associate with each vector A_n in $[n;\ x_1,\ x_2,\ \ldots,\ x_k]$ the vector of $(n-1)$ elements $P(A_n)$ obtained as follows:

(i) P replaces every element $a_i \leqq k$ of A_n by zero. (As $a_1 \leqq k$ by (2.5.1), the first element is always replaced by zero.)

(ii) Replace every element $a_i > k$ of A_n, by $a_i - k$.

(iii) Suppress the first zero element in A_n leaving a vector of $(n-1)$ elements.

We assert that P is a one-one mapping from $[n; x_1, x_2, \ldots, x_k]$ $(n \geqq 2)$ onto

$$\sum = \bigcup_{y_1=0}^{x_1} \cdots \bigcup_{y_k=0}^{x_k} [n-1;\ y_1, \ldots, y_k] .$$

A moment's reflection shows that P maps $[n;\ x_1, x_2, \ldots, x_k]$ into Σ. We now show that P is one to one.

Consider the vectors in the set $[n;\ x_1,\ x_2,\ \ldots,\ x_k]$ partitioned into two parts

(i) the first part consisting of elements $(a_1,\ a_2,\ \ldots,\ a_r)$ where $a_r \leqq k$, and

(ii) the last part $a_{r+1}, a_{r+2}, \ldots, a_n$ where $a_{r+1} > k$.

We remark first that the last part may be empty in which case all the elements are replaced by zero elements by P and the vector is mapped into $(\underbrace{0,\ 0,\ \ldots,\ 0}_{n-1})$. Let A_n, B_n be distinct vectors in $[n;\ x_1,\ \ldots,\ x_k]$. Then A_n, B_n may differ in one of the following ways.

(1) A_n, B_n may have different number of elements $\leqq k$ in which case $P(A_n)$, $P(B_n)$ will have different number of zero elements and $P(A_n) \neq P(B_n)$.

(2) A_n, B_n may have the same number of elements $\leqq k$ but differ in the last part. In this case, since P subtracts k from all elements $> k$, $P(A_n)$ and $P(B_n)$ must be different.

(3) A_n, B_n may have the same number of elements $\leqq k$ but do not differ in the last part i.e. A_n, B_n differ in the first part since they were assumed distinct. In this case, since A_n, B_n belong to $[n; x_1,\ldots,x_k]$, a little consideration shows that it cannot happen that A_n, B_n differ in the first part but not in the last. Hence this case cannot occur.

Clearly our cases (a),(b),(c) cover all possibilities for A_n, B_n. Thus we have seen that whenever A_n, B_n are distinct elements of $[n; x_1,x_2,\ldots,x_k]$, $P(A_n)$, $P(B_n)$ are also distinct. Hence P is a one-one into mapping.

A little consideration now shows that P is in fact one-one onto. The inverse mapping of P is easily constructed. Indeed, the mapping Q defined below is the desired inverse. For $n \geqq 2$, associate with each vector σ_{n-1} in Σ the vector of n elements $Q(\sigma_{n-1})$ obtained as follows:

(i) Add a zero first element to σ_{n-1} leaving a vector of n elements.

(ii) Add k to every non-zero element.

(iii) If there are $(j+1)$ zero elements in the vector of n elements obtained in (i) and (ii) choose

$$a_{j+1} = a_j = \ldots\ldots = a_{j-(x_k-y_k)+2} = k$$

$$a_{j+1-(x_k-y_k)} = \ldots\ldots = a_{j+1-(x_k-y_k)-(x_{k-1}-y_{k-1})+1} = k-1$$

etc. i.e. replace the last x_k-y_k zero elements by k, the next $x_{k-1}-y_{k-1}$ by $k-1$, and so on.

It is easily verified that Q is a one-one mapping from Σ into $[n; x_1, x_2, \ldots, x_k]$. This concludes the proof of the lemma.

Illustration: Consider the subset [3; 1,1,1] of $A_{3,3}$. It has the following elements

123, 126, 129, 135, 138, 156, 159, 168, 234, 237, 246, 249, 267, 345, 348, 357

Table II shows the results of applying P to the elements of [3; 1,1,1]. The headings (i), (ii), (iii) refer to the corresponding operations defined on page 48 in describing P.

Table II

A_3	(i)	(ii)	(iii)	$P(A_3)$ belongs to
123	000	000	00	[2;0,0,0]
126	006	003	03	[2;0,0,1]
129	009	006	06	[2;0,0,1]
135	005	002	02	[2;0,1,0]
138	008	005	05	[2;0,1,0]
156	056	023	23	[2;0,1,1]
159	059	026	26	[2;0,1,1]

A_3	(i)	(ii)	(iii)	$P(A_3)$ belongs to
168	068	035	35	[2;0,1,1]
234	004	001	01	[2;1,0,0]
237	007	004	04	[2;1,0,0]
246	046	013	13	[2;1,0,1]
249	049	016	16	[2;1,0,1]
267	067	034	34	[2;1,0,1]
345	045	012	12	[2;1,1,0]
348	048	015	15	[2;1,1,0]
357	057	024	24	[2;1,1,0]

Similarly the result of applying Q to some elements of Σ are contained in Table III, where the headings (i), (ii), (iii) refer to the definition of Q .

Table III

σ_2	(i)	(ii)	(iii)
00	000	000	123
04	004	007	237
13	013	046	246
23	023	056	156

The table may easily be completed. Thus we see that Q is indeed the inverse mapping of P as asserted.

§ 2.4 A refinement of the ballot theorem.

In this section we refine the ballot theorem using the analogue of the multinomial theorem discussed in section 2.2. More precisely, we will prove our main theorem which yields the ballot theorem as a special case.

We shall refine the ballot theorem for all integral $k \geqq 2$ using the lattice path interpretation of section 2.3. Our refinement of the ballot theorem does not apply for the case $k = 1$ as will be evident from our arguments. More precisely when $k = 1$, our methods will yield the ballot theorem itself, but no true refinements of it.

Lemma 2.3.1 generalises the ballot theorem for all points (m, n), where $n \geqq 2$ and $m \equiv 1 \pmod k$, $m > kn$. The proof of our main theorem will now be completed, apart from the obvious case of points $(m,1)$, $(m,0)$, by using a rather similar argument for the cases $m \equiv j \pmod k$, $j = 2,3,\ldots,k$. For simplicity we prove the theorem for $k = 3$. The proof for any integral $k \geqq 3$ is analogous.

Consider the <u>positive</u> elements $(a_j, \ldots, a_n)$ of a path vector to (m, n). If <u>exactly</u> x_1, x_2, x_3 of these positive elements are $\equiv i \pmod 3$, $i = 1, 2, 3$ respectively, we say that the path belongs to $[m,n; x_1,x_2,x_3]$. Clearly $x_1 + x_2 + x_3 = n - j + 1 \leqq n$, and x_1, x_2, x_3 are non-negative integers. Let $(m,n; x_1, x_2, x_3)$ denote the number of vectors (paths) in $[m,n; x_1, x_2, x_3]$. Then we state our main theorem as

Theorem 2.4.1:

If $x_1 + x_2 + x_3 = n$, then

$$(2.4.1) \qquad (m,n;x_1,x_2,x_3) = \begin{cases} (n;\ x_1,\ x_2,\ x_3) & \text{if } m = 3n+4 \\ (n;\ x_2,\ x_3,\ x_1^*) & \text{if } m = 3n+5 \\ (n;\ x_3,\ x_1^*,\ x_2^*) & \text{if } m = 3n+6 \end{cases}$$

$$(2.4.1') \qquad (3n+2,n;x_1,x_2,x_3) = (n-1;x_2,x_3,(x_1-1)^*)$$

$$(2.4.1'') \qquad (3n+3,n;x_1,x_2,x_3) = (n-1;x_3,x_1,(x_2-1)^*) + (n-1;x_3,(x_1-1)^*,x_2^*) \quad .$$

For all other points with $m > 3n$ the result may be obtained as a special case of (2.4.1), (2.4.1') or (2.4.1"). (A more compact statement of this theorem is given in (2.5.4). Note, however, that the asterisks in (2.4.1) appear in a cyclic order in these equations. We will give an interesting application of it in section 2.6.)

Proof:

Lemma 2.3.1 proves the theorem for the points $(3n+4,n)$.

To prove the theorem for $m = 3n+5$, we note that every path to $(3n+5,n)$ with exactly n positive elements in its path vector must intersect the line $x = 3n+4$ <u>horizontally</u> at some point $(3n+4,r)$ where $r \leqq n$. Now all paths in $[3n+5, x_1 + x_2 + x_3 = n;\ x_1,\ x_2,\ x_3]$ must pass through $P_n = (3n+4,n)$ and their last steps must be horizontal and of unit length,

namely, the line joining P_n and $(3n+5,n)$. Also all paths to P_n have been classified into sets $[3n+4,n;y_1, y_2, y_3]$ with $y_1 + y_2 + y_3 \leqq n$. Hence every path in $[3n+4,n; y_1, y_2, y_3]$ when viewed as a path to $(3n+5,n)$ by joining P_n to the point $(3n+5,n)$ will belong to the class $[3n+5,n; y_3 + (n-\Sigma y_i), y_1, y_2]$. Thus, in order that a path to P_n when extended one unit horizontally to $(3n+5,n)$ belong to the class $[3n+5,n = x_1 + x_2 + x_3; x_1, x_2, x_3]$ it is necessary and sufficient that

$$y_1 = x_2 , \; y_2 = x_3 \; \text{ and } \; y_3 + (n - \sum_i y_i) = x_1 .$$

Here $(n-\Sigma y_i) = x_1 - y_3$ is non-negative, since it represents the number of zero elements in any path vector belonging to $[3n+4,n; y_1, y_2, y_3]$. Clearly these zero elements transform to 1's when the path is extended to $(3n+5,n)$. Hence the number of paths to $(3n+5,n)$ which have exactly n positive elements in their path vectors is given by

$$\sum_{y_3=0}^{x_1} (3n+4,n;x_2,x_3,y_3) = \sum_{y_3=0}^{x_1} (n;x_2,x_3,y_3) = (n;x_2,x_3,x_1^*) .$$

A similar proof may be given for $m = 3n+6$. To prove (2.4.1'), we first note that whenever $x_1 = 0$, the right hand side of the equation vanishes. Now every path to $(3n+2,n)$ which has exactly n positive elements in its path vector must pass through $(3n+1,n-1)$. Thus every path in $[3n+2,n = x_1 + x_2 + x_3; x_1, x_2, x_3]$ may be shortened one unit horizontally and then one unit vertically to yield a path in

$$\bigcup_{y_1=0}^{x_1-1} [n-1; x_2, x_3, y_1] .$$

Hence, whenever $\sum_i x_i = n$, we have as before,

$$(3n+2,n;x_1,x_2,x_3) = \sum_{y_1=0}^{x_1-1} (n-1;x_2,x_3,y_1) = (n-1;x_2,x_3,(x_1-1)^*) .$$

To prove (2.4.1") we only have to note that all paths to $(3n+3,n)$ having exactly n positive elements in their path vectors can be partitioned into paths which pass through $(3n+1,n)$ and $(3n+2,n-1)$. Clearly the number of paths passing through $(3n+1,n)$, which when extended, yield paths in $[3n+3,n;\ x_1,\ x_2,\ x_3]$ is

$$(3n + 2,\ n;\ x_2,\ x_3,\ x_1) = (n-1;\ x_3,\ x_1,\ (x_2-1)*)$$

using (2.4.1'). By an argument similar to the one used in proving (2.4.1) it can be seen that the number of paths passing through $(3n+2,n-1)$, which when extended, yield paths in $[3n+3,n;\ x_1,\ x_2,\ x_3]$ is

$$\sum_{y_1=0}^{x_1-1} (3n+2,n-1;x_2,x_3,y_1) = (3n+2,n-1;x_2,x_3,(x_1-1)*) = (n-1;x_3,(x_1-1)*,x_2^*)$$

using (2.4.1). Hence the total number of paths to $(3n+3,n)$ is given by (2.4.1").

To complete the proof we note that any point (m,n) with $m > 3n$ which does not fall into the previous cases may be represented as $(3n+4,t)$, $(3n+5,t)$ or $(3n+6,t)$ where $t < n$. Excluding, for the moment, the rather obvious cases $t = 0,1$, let us first consider points $(3n+4,t)$. Clearly

$[3n+4, t;\ x_1,\ x_2,\ x_3]$ with $x_1 + x_2 + x_3 = t$ must equal $[3n+4, n; x_1, x_2, x_3]$, and hence from Lemma 2.3.1

$$(3n+4, t; x_1, x_2, x_3) = (n; x_1, x_2, x_3) \quad \text{where} \quad \sum_i x_i = t < n \quad .$$

Again by a repetition of the argument used in proving (2.4.1) we have, whenever $\sum_i x_i = t < n$,

$$(3n+5, t; x_1, x_2, x_3) = (3n+5, n; x_1, x_2, x_3) = \sum_{y_1=o}^{x_1} (3n+4, t; x_2, x_3, y_1)$$

$$= (n; x_2, x_3, x_1^*) \quad .$$

The remaining case is similarly treated.

Thus (2.4.1) is true even if $\sum_i x_i = t < n$ and all other cases $(3n+j, t)$, $t < n$, $j = 4,\ 5,\ 6$ follow from the obvious identities

$$(m, n; x_1, x_2, x_3) = (m, n-1; x_1, x_2, x_3) = \ .\ .\ .\ .\ = (m, t; x_1, x_2, x_3)$$

where $\sum_i x_i = t < n$.

In the next section we will derive the ballot theorem as a special case of Theorem 2.4.1. It will then be seen that we are not only able to give the number of paths to any point (m,n) but to all points $(m,n-i)$ $(i=1,\ldots,n)$ and that we are also able to classify these paths according to congruences of the elements of path vectors.

§ 2.5 Derivation of the ballot theorem

Before we derive the ballot theorem, we first extend the identity

(2.2.7) to cases in which $\sum_i x_i = n+1$. We recall that in (2.2.2) we have set

$$(n;\ x_1,x_2,\ldots,x_k) = 0 \qquad \text{if } \sum_i x_i > n$$

(and thus, in particular, if $\sum_i x_i = n+1$.)

Hence for $\sum_i x_i = n+1$, any expression (with one asterisk)

$$(n;x_1,\ldots,x_j^*,\ldots,x_k) = \sum_{y_j=o}^{x_j} (n;x_1,\ldots,y_j,\ldots,x_k)$$

$$= \sum_{y_j=o}^{x_j-1} (n;x_1,\ldots,y_j,\ldots,x_k)$$

or

(2.5.1) $$(n;x_1,\ldots,x_j^*,\ldots,x_k) = (n;x_1,x_2,\ldots,(x_j-1)^*,\ldots,x_k)$$

is well defined.

Consider next a sum

(2.5.2) $$\sum_{y_i=o}^{x_i} \sum_{y_j=o}^{x_j} (n;\ x_1,\ \ldots,\ y_i,\ \ldots,\ y_j,\ldots,\ x_k)$$

with $\sum_i x_i = n+1$. This sum may be split as

$$\sum_{y_i=o}^{x_i-1} \sum_{y_j=o}^{x_j} (n;x_1,\ldots,y_i,\ldots,y_j,\ldots,x_k) + \sum_{y_j=o}^{x_j} (n;x_1,\ldots,x_i,\ldots,y_j,\ldots,x_k)$$

Now using (2.5.1) to simplify the last sum, the sum in (2.5.2) equals

$$(2.5.3) \quad (n;x_1,\ldots,(x_i-1)^*,\ldots,x_j^*,\ldots,x_k) + (n;x_1,\ldots,x_i,\ldots,(x_j-1)^*,\ldots,x_k).$$

Since the finite summation in (2.5.2) also equals

$\sum\limits_{y_j=0}^{x_j} \sum\limits_{y_i=0}^{x_i} (n;x_1,\ldots,y_i,\ldots,y_j,\ldots,x_k)$ which may be split as

$$\sum_{y_j=0}^{x_j-1} \sum_{y_i=0}^{x_i} (n;x_1,\ldots,y_i,\ldots,y_j,\ldots,x_k) + \sum_{y_i=0}^{x_i} (n;x_1,\ldots,y_i,\ldots,x_j,\ldots,x_k)$$

an expression analogous to (2.5.3), namely,

$$(2.5.3') \quad (n;x_1,\ldots,x_i^*,\ldots,(x_j-1)^*,\ldots,x_k) + (n;x_1,\ldots,x_i^*,\ldots,x_j,\ldots,x_k)$$

must equal (2.5.3) and we take the common value of (2.5.3) and (2.5.3') as $(n;x_1,\ldots,x_i^*,\ldots,x_j^*,\ldots,x_k)$.

In general, if $[i_1,i_2,\ldots,i_r]$ is a subset of $[1,2,\ldots,k]$ with $1 \leq i_1 < i_2 < \ldots < i_r \leq k$, then $(n; x_1,\ldots,x_{i_1}^*,\ldots,x_{i_r}^*,\ldots,x_k)$ for $\sum\limits_i x_i = n+1$, may be taken as any of $r!$ equal expressions and is well defined. This completes the extension of our definition of the left hand side of (2.2.7) when $\sum\limits_i x_i = n+1$.

We also note that equations (2.4.1'), (2.4.1") which are valid for $\Sigma x_i = n$, may now be written as

$$(3n+2,n;x_1,x_2,x_3) = (n-1;x_2,x_3,x_1^*)$$

and

$$(3n+3,n;x_1,x_2,x_3) = (n-1;x_3,x_1^*,x_2^*) \quad .$$

Hence our main theorem takes on the following simple form:

If $\Sigma x_i = t \leqq n$, then

$$(2.5.4) \qquad (m,t;x_1,x_2,x_3) = \begin{cases} (n;x_1,x_2,x_3) & \text{if } m = 3n+4 \\ (n;x_2,x_3,x_1^*) & \text{if } m = 3n+5 \\ (n;x_3,x_1^*,x_2^*) & \text{if } m = 3n+6 \quad . \end{cases}$$

For points $(3n+1,n)$, $(3n+2,n)$ and $(3n+3,n)$ a similar result holds with n replaced by $(n-1)$, $n \geqq 1$. (Clearly the number of paths to $(m,0)$ is 1.)

In the form (2.5.4), the generalization of the ballot theorem is straightforward for any $k \geqq 2$.

We now derive the ballot theorem as a special case of (2.5.4). We restrict ourselves to the typical point $(3n+5,t)$, where $t \leqq n$ is a positive integer, as other cases are similar.

Now paths to $(3n+5,t)$ from the origin can be partitioned into $(t+1)$ classes, according to the number of positive elements $c(=0,1,2,\dots,t)$

in their path vectors. A path which has exactly c positive elements belongs (for appropriate x_1, x_2, x_3) to $[3n+5, t; x_1, x_2, x_3]$ with $\Sigma x_i = c$. Thus the number of paths to $(3n+5,t)$ with exactly c positive elements is $\sum_{x_1+x_2+x_3=c} (n; x_2, x_3, x_1^*)$ from (2.5.4). Hence the total number of paths to $(3n+5,t)$ is $\sum_{c=o}^{t} \left\{ \sum_{x_1+x_2+x_3=c} (n; x_2, x_3, x_1^*) \right\}$. The quantity in braces is

$$\sum_{\substack{\Sigma x_i = c \\ i}} \binom{n+x_1+1}{x_1} \binom{n+x_2}{x_2} \binom{n+x_3}{x_3} \left[1 - \frac{x_1}{n+2} - \frac{x_2+x_3}{n+1} \right]$$

$$= \sum_{r=o}^{c} \binom{n+r+1}{r} \sum_{x_2+x_3=c-r} \binom{n+x_2}{x_2} \binom{n+x_3}{x_3} \left[1 - \frac{r}{n+2} - \frac{c-r}{n+1} \right]$$

$$= \sum_{r=o}^{c} \binom{n+r+1}{r} \left[\text{coefficient of } x^{c-r} \text{ in } (1-x)^{-(2n+2)} \right] \cdot \left[1 - \frac{r}{n+2} - \frac{c-r}{n+1} \right]$$

$$= \sum_{r=o}^{c} \binom{n+r+1}{r} \binom{2(n+1)+c-r-1}{c-r} \left[1 - \frac{r}{n+2} - \frac{c-r}{n+1} \right]$$

$$= \binom{3n+3+c}{c} - 3 \binom{3n+3+c}{c-1} .$$

The last step is obtained by using the well known combinatorial identity

$$\sum_{j=o}^{k} \binom{a+k-j-1}{k-j}\binom{b+j-1}{j} = \binom{a+b+k-1}{k} .$$

Thus the total number of paths to $(3n+5,t)$ is

$$\sum_{c=o}^{t} \left\{ \binom{3n+3+c}{c} - 3\binom{3n+3+c}{c-1} \right\} = \frac{3n+5-3t}{3n+5+t} \binom{3n+5+t}{t}$$

using the identity I mentioned in section 2.2.

This completes the derivation of the ballot theorem.

§ 2.6 Some applications

In this section we will first prove a simple theorem suggested by the definition of $(n;x_1, x_2, \ldots, x_k)$ of section 2.2. Then we will derive the "one A.P." case of Mohanti and Narayana discussed in [11].

Let n, x_i $(i = 1, 2, \ldots, k)$ be non-negative integers. For $n \geqq 1$, consider the function defined as follows:

$$(0;\ 0,\ 0,\ \ldots,\ 0)^- = 1$$

$$(2.6.1) \quad (n;x_1,x_2,\ldots,x_k)^- = \begin{cases} 0 & \text{if } \sum\limits_i x_i > n \\ \sum\limits_{y_1=o}^{x_1} \cdots \sum\limits_{y_k=o}^{x_k} (-1)^{\Sigma(x_i-y_i)} (n-1;y_1,\ldots,y_k)^-, & \text{otherwise} . \end{cases}$$

Then we have the

Theorem 2.6.1:

With $\Sigma x_i \leqq n$,

$$(2.6.2) \qquad (n;x_1,x_2,\ldots,x_k)^- = (-1)^{\Sigma x_i} \prod_{i=1}^{k} \binom{n+1}{x_i} \left[1 - \frac{\sum_i x_i}{n+1}\right] .$$

Proof:

For $n = 0, 1$ it is easily checked that (2.6.2) holds.

Suppose then that (2.6.2) holds for $n = m-1$ $(m \geqq 2)$ and $\sum_i x_i \leqq m-1$, we will show that (2.6.2) holds for $n = m$ and $\sum_i x_i \leqq m$. Now, by (2.6.1)

$$(m;x_1,x_2,\ldots,x_k)^- = \sum_{y_1=o}^{x_1} \cdots \sum_{y_k=o}^{x_k} (-1)^{\sum_i (x_i-y_i)} (m-1;y_1,\ldots,y_k)^- .$$

Since $\sum_i x_i \leqq m$, we have $\sum_i y_i < m$ unless $y_1 = x_1,\ldots,y_k = x_k$ and $x_1 + \ldots + x_k = m$. This is a single term which vanishes by (2.6.1). Hence we can assume that $\sum_i y_i \leqq m-1$, in which case we can apply (2.6.2) to obtain

$$(m;x_1,x_2,\ldots,x_k)^- = \sum_{y_1=o}^{x_1} \cdots \sum_{y_k=o}^{x_k} (-1)^{\sum_i (x_i-y_i)} (-1)^{\sum_i y_i} \prod_{i=1}^{k} \binom{m-1+y_i}{y_i} \left[1 - \frac{\sum_i y_i}{m}\right]$$

$$= (-1)^{\sum_i x_i} \left\{ \sum_{y_1=o}^{x_1} \cdots \sum_{y_k=o}^{x_k} \prod_{i=1}^{k} \binom{m-1+y_i}{y_i} \left[1 - \frac{\sum_i y_i}{m}\right] \right\}$$

Now, using an argument similar to the one used in proving Theorem 2.2.1 we obtain

$$(m;x_1,\ldots,x_k)^- = (-1)^{\sum_i x_i} \prod_{i=1}^{k} \binom{m+x_i}{x_i} \left[1 - \frac{\sum_i x_i}{m+1}\right] .$$

This completes the proof.

Equations (2.6.1) and (2.6.2) suggest that by proceeding similarly to (2.6.2) we can establish the general identity

(2.6.3) $$(n;x_1,\ldots,x_{i_1}^*,\ldots,x_{i_r}^*,\ldots,x_k)^- =$$

$$= \sum_{y_{i_1}=o}^{x_{i_1}} \cdots \sum_{y_{i_r}=o}^{x_{i_r}} (-1)^{\sum_{j=1}^{r}(x_{i_j}-y_{i_j})} (n;x_1,\ldots,y_{i_1},\ldots,y_{i_r},\ldots,x_k)^-$$

$$= (-1)^{\sum_i x_i} \prod_{j=1}^{r} \binom{n+x_{i_j}+1}{x_{i_j}} \prod_{j\epsilon S} \binom{n+x_j}{x_j} \left[1 - \frac{s^*}{n+2} - \frac{s}{n+1}\right] .$$

The identity (2.6.3) is analogous to (2.2.7) and the notations used are the same as in (2.2.7).

Next, we direct our attention to deriving some preliminary results to be used in the derivation of the one A.P. case, to be discussed a little later. Consider the set $A_r(n,k)$ of vectors of non-negative integers $(a_1,a_2,\ldots,a_n)$ whose elements satisfy

(2.6.4) $$\begin{cases} 0 \leq a_1 \leq a_2 \leq \cdots \leq a_n \\ a_i \leq ki + r \qquad i = 1, 2, \ldots, n; \quad r \text{ is} \\ \text{an integer satisfying } 0 \leqq r \leqq k . \end{cases}$$

Now introduce the subsets $A_r(n;x_1,\ldots,x_k)$ of $A_r(n,k)$ such that every vector in $A_r(n;x_1,\ldots,x_k)$ has exactly x_i of its positive elements $\equiv i \pmod k$ $i = 1,2,\ldots,k$ respectively. Let $A_r^*(n;x_1,\ldots,x_k)$ denote the number of vectors in $A_r(n;x_1,\ldots,x_k)$. Then we have the

Theorem 2.6.2:

With $\sum_i x_i \leqq n$,

$$A_r^*(n;x_1,\ldots,x_k) = (n;x_{r+1},\ldots,x_k,x_1^*,\ldots,x_r^*) \tag{2.6.5}$$

where the expression on the right hand side of (2.6.5) is defined by the identity (2.2.7).

We first interpret (2.6.5). Essentially the theorem states that to obtain the number of vectors in $A_s(n;x_1,\ldots,x_k)$, $0 \leqq s \leqq k$, we only have to permute cyclically $x_1,\ldots,x_k$ in $(n;x_1,\ldots,x_k)$ s times and then place asterisks on $x_1,\ldots,x_s$. Note that when $s = 0$, the sets $A_s(n,k)$ and $A_{n,k}$ (defined in (2.3.1)) coincide and so do $A_s^*(n;x_1,\ldots,x_k)$ and $(n;x_1,\ldots,x_k)$. Similarly, when $s = k$ we have

$$A_k^*(n;x_1,\ldots,x_k) = (n;x_1^*,\ldots,x_k^*) = (n+1;x_1,\ldots,x_k) \quad . \tag{2.6.6}$$

Proof:

It is convenient to prove the theorem for $r = 1$. The proof will then be completed for any r, $0 < r < k$, by essentially repeating the same argument. For $n = 0$, the theorem is trivially true. For $n \geqq 1$, associate with each vector A_n in $A_1(n;x_1,\ldots,x_k)$ the vector of n elements $R(A_n)$ obtained by subtracting 1 from

every <u>positive</u> element of A_n. We assert that R is a one-one mapping from $A_1(n;x_1,\ldots,x_k)$ <u>onto</u> $\bigcup_{y_1=o}^{x_1} [n;x_2,\ldots,x_k,y_1]$. To see this, first note that under R the elements of A_n which are $\equiv i(\text{mod } k)$, $i = 2,\ldots,k$ are mapped into elements $\equiv i-1(\text{mod } k)$, $i = 2,\ldots,k$ respectively and the elements $\equiv 1(\text{mod } k)$ are mapped either into elements $\equiv k(\text{mod } k)$ or into zero. Thus R is an into mapping. A little consideration now shows that R is one-one and the fact that R is onto follows by using an argument analogous to the one used in proving Lemma 2.3.1. Thus the number of elements in the two sets is equal and equation (2.6.5) holds for $r = 1$.

Next consider the subset $A_2(n;x_1,\ldots,x_k)$ of $A_2(n,k)$. The application of R to the elements of $A_2(n;x_1,\ldots,x_k)$ subtracts 1 from every positive element and the elements now satisfy

$$\begin{cases} 0 \leqq a_1 \leqq \cdots \leqq a_n \\ a_i \leqq ki+i \qquad i = 1,2,\ldots,n \quad . \end{cases}$$

A little consideration shows that there exists a one to one correspondence between the vectors in $A_2(n;x_1,\ldots,x_k)$ and $\bigcup_{y_1=o}^{x_1} A_1(n;x_2,\ldots,x_k,y_1)$. Thus

$$A_2^*(n;x_1,\ldots,x_k) = \sum_{y_1=o}^{x_1} A_1^*(n;x_2,\ldots,x_k,y_1)$$

$$= \sum_{y_1=o}^{x_1} (n;x_3,\ldots,x_k,y_1,x_2^*)$$

$$= (n; x_3, \ldots, x_k, x_1^*, x_2^*) \quad .$$

It is now clear how the proof may be completed in r steps.

Illustration: Consider the subset $A_2(2;1,1,0)$ of $A_2(2,3)$ whose elements are 12, 15, 18, 24, 27, 45, 48, 57. Then

$$R(A_2(2;1,1,0)) = [01,\ 04,\ 07,\ 13,\ 16,\ 34,\ 37,\ 46]$$

$$= \bigcup_{y_1=0}^{1} A_1(2;1,0,y_1)$$

$$= \bigcup_{y_1=0}^{1} \bigcup_{y_2=0}^{1} [2;0,y_1,y_2]$$

and we have,

$$A_2^*(2;1,1,0) = 8$$

$$(2;0,1^*,1^*) = 8 \quad .$$

Finally, we make the following remarks:

(i) Consider the set of vectors whose elements are non-negative integers $(a_1, \ldots, a_n)$ satisfying

$$(2.6.7) \qquad \begin{cases} 0 \leqq a_1 \leqq \cdots \leqq a_n \\ a_i \leqq k(i+p), \quad i = 1, 2, \ldots, n; \ p \geq 0 \text{ is an integer.} \end{cases}$$

Then $(n+p; x_1, \ldots, x_k)$, $\sum_i x_i \leqq n$, gives the number of vectors in the subset of (2.6.7) which is such that exactly x_i <u>positive</u> elements of every vector contained in it are $\equiv i \pmod k$, $i = 1, 2, \ldots, k$ respectively.

(ii) Next, consider the set of vectors whose elements are non-negative integers $(a_1, a_2, \ldots, a_n)$ satisfying

$$(2.6.8) \qquad \begin{cases} (a) \quad 0 \leqq a_1 \leqq a_2 \leqq \cdots \leqq a_n \\ (b) \quad a_i = 0 \quad \text{for } i = 1, 2, \ldots, q \ ; \quad q < n \\ (c) \quad 0 \leqq a_i \leqq k(i-q) \quad \text{for } i = q+1, \ldots, n \ . \end{cases}$$

Then $(n-q; x_1, x_2, \ldots, x_k)$, $\sum_i x_i \leqq n-q$, gives the number of vectors in the subset of (2.6.8) which is such that exactly x_i <u>positive</u> elements of every vector contained in it are $\equiv i \pmod k$, $i = 1, 2, \ldots, k$, respectively.

We are now in a position to solve the "one A.P." case. Let $A_n(a,b)$ denote the set of vectors $(a_1, a_2, \ldots, a_n)$ such that

(2.6.9)
$$\begin{cases} 0 \leqq a_1 \leqq \cdots \leqq a_n \\ \\ 0 \leqq a_i \leqq a + (i-1)b \ , \quad i = 1,2,\ldots,n \ ; \ a,b \ \text{ are integers} \geqq 0 \ . \end{cases}$$

Then we have the

Theorem 2.6.3:

The number of vectors in $A_n(a,b)$ is given by

$$A_n^*(a,b) = \frac{a+1}{a+1+n(b+1)} \binom{a+1+n(b+1)}{n} \ .$$

Proof:

Let $c, d < b$ be non-negative integers such that

(2.6.10) $$a = bc + d \ .$$

Then (2.6.9) may be re-written as

(2.6.11)
$$\begin{cases} 0 \leq a_1 \leq \cdots \leq a_n \\ \\ 0 \leq a_i \leq d + (c+i-1)b \quad i = 1,\ldots,n \end{cases}$$

Partition $A_n(a,b)$ into subsets $A_n(a,b;x_1,\ldots,x_b)$ such that every vector in $A_n(a,b;x_1,\ldots,x_b)$ has exactly x_i of its positive elements $\equiv i \pmod b$, $i = 1, 2, \ldots, b$ respectively. If $A_n^*(a,b;x_1,\ldots,x_b)$ denotes the number of vectors in $A_n(a,b;x_1,\ldots,x_b)$ then by preceding remarks and Theorem 2.6.2 we have

$$A_n^*(a,b;x_1,\ldots,x_b) = (n+c-1;x_{d+1},\ldots,x_b,x_1^*,\ldots,x_d^*)$$

and hence

$$A_n^*(a,b) = \sum_{x_1+\cdots+x_b \leq n}' A_n^*(a,b;x_1,\ldots,x_b)$$

$$= \sum_{\sum_i x_i \leq n}' \prod_{i=1}^{d} \binom{n+c+x_i}{x_i} \prod_{j=d+1}^{b} \binom{n+c-1+x_j}{x_j} \left[1 - \frac{\sum_i x_i}{n+c+1} - \frac{\sum_j x_j}{n+c}\right]$$

Letting $\sum_i x_i = r$, $\sum_j x_j = s$ and $\sum_i x_i + \sum_j x_j = t$, say, we have

$$A_n^*(a,b) = \sum_{t=o}^{n} \left\{ \sum_{\sum_i x_i + \sum_j x_j = t}' \prod_{i=1}^{d} \binom{n+c+x_i}{x_i} \prod_{j=d+1}^{b} \binom{n+c-1+x_j}{x_j} \times \right.$$

$$\left. \times \left[1 - \frac{\sum_i x_i}{n+c+1} - \frac{\sum_j x_j}{n+c}\right]\right\}$$

$$= \sum_{t=o}^{n} \left\{ \sum_{r=o}^{t} \binom{(n+c+1)d+r-1}{r} \binom{(n+c)(b-d)+t-r-1}{t-r} \left[1 - \frac{r}{n+c+1} - \frac{t-r}{n+c}\right] \right\}$$

$$= \sum_{t=o}^{n} \left\{ \binom{(n+c)b+d+t-1}{t} - b \binom{(n+c)b+d+t-1}{t-1} \right\}$$

$$= \frac{a+1}{a+1+n(b+1)} \binom{a+1+n(b+1)}{n}$$

which proves the assertion.

Note that our approach not only yields $A_n^*(a,b)$ but also classifies the vectors of $A_n^*(a,b)$ according to congruences of their elements.

CHAPTER III

A VARIETY OF COMBINATORIAL RESULTS.

§ 3.1 Introduction and some preliminary remarks

In the last chapter we discussed in detail an analogue of the multinomial theorem and its connection with a certain set $A_{n,k}$ of vectors of non-negative integers. A classification of the vectors in the set $A_{n,k}$ according to the congruence properties of their elelemnts led us naturally to a refinement of the ballot theorem. We take this occasion to remark that there is some similarity between the "main" Theorem 2.4.1 and the theorems of Chapter I although our approach is entirely different.

Another very interesting application of the function $(n;x_1,x_2,\ldots,x_k)$ is for the special case $k = 2$ when the set $A_{n,2}$ and the set of simple sampling plans of size n in the plane may be shown to be isomorphic partially ordered sets. We do not propose to discuss, apart from a brief mention in this section, any properties of simple sampling plans. A detailed discussion of such properties as well as enumeration problems concerning simple sampling plans of size n is presented in [10] , and we also refer to [12] where sampling plans were first introduced formally.

A closer examination of certain properties of simple sampling plans suggests yet another source of recursions of the type discussed in Chapter II. Such a study has already been undertaken in [13] where a subclass of all sampling plans called 'regular' sampling plans has been discussed. Curiously enough the only other non-trivial function analogous

to the multinomial that we have been able to find has the same relation with regular sampling plans as the function $(n;\ x_1,\ldots,x_k)$ has with all simple sampling plans. However, in what follows we are only interested in the combinatorial aspects of these analogues. With this end in view, we give a brief account of the function $\{n;x_1,\ldots,x_k\}$ arising out of the regular sampling plans in the following section. We also study some other types of recursions in an attempt to find out any other non-trivial functions (analogous to the multinomial) for which an explicit expression may be given.

The rest of this chapter is devoted to certain special topics. The diverse nature of these special topics prevents us from describing them here with any degree of precision. Thus the connection between these topics and our general approach will only be discussed, when appropriate, in the following sections.

§ 3.2 Further analogues

This section is devoted to the discussion of some recursive functions in an attempt to find some non-trivial functions analogous to the multinomial and having explicit expressions. The functions defined by these recursions can be identified with certain subsets of the set of vectors $A_{n,k}$ and also have connections with the simple sampling plans which, in view of our remarks in section 3.1, we will not discuss here. A lattice path interpretation can be easily given to these subsets of vectors.

We first consider the function $\{n;x_1,\ldots,x_k\}$ which is the only other non-trivial analogue of the multinomial theorem that we have been able

to find. In the course of our discussion of $\{n;x_1,\ldots,x_k\}$ we will see that its applications are analogous to the applications of $(n;x_1,\ldots,x_k)$. We remark that some applications of $\{n;x_1,\ldots,x_k\}$ that we give below are also contained in [13] but our approach seems a little simpler and our results are also more comprehensive.

Let n, x_i, δ_i $(i = 1, 2, \ldots, k)$ be non-negative integers. For $n \geqq 1$, consider the function defined recursively as follows:

$$\{0;\, 0,\, 0,\, \ldots,\, 0\} = 1$$

$$(3.2.1)\qquad \{n;x_1,x_2,\ldots,x_k\} = \begin{cases} 0 & \text{if } \sum_i x_i > n \\ \displaystyle\sum_{\substack{\delta_i = o \text{ or } 1 \\ i=1,2,\ldots,k}} \{n-1;x_1-\delta_1,x_2-\delta_2,\ldots,x_k-\delta_k\} & \text{otherwise} \end{cases}.$$

Here the 'Σ' sign indicates that the second right-hand member of (3.2.1) is the sum of 2^k terms. The function $\{n;x_1,\ldots,x_k\}$ is given explicitly for $\sum_i x_i \leqq n$ by Theorem 3.2.1 which is proved by an argument similar to the one used in proving Theorem 2.2.1.

Theorem 3.2.1:

When $\sum_i x_i > n$,

$$\{n;x_1,\ldots,x_k\} = 0 ,$$

while for $\sum_i x_i \leqq n$,

$$\{n;x_1,x_2,\ldots,x_k\} = \prod_{i=1}^{k} \binom{n+1}{x_i} \left[1 - \frac{\sum_i x_i}{n+1}\right] . \tag{3.2.2}$$

Proof:

The first statement is trivially true by <u>definition</u>. To prove (3.2.2) we proceed by induction. For $n = 0, 1$ it is easily checked that (3.2.2) holds.

Assume then that (3.2.2) holds for all positive integers $n \leqq m$ $(m \geqq 1)$. Then, with $\sum_i x_i \leqq m+1$, we have

$$\{m+1;x_1,x_2,\ldots,x_k\} = \sum_{\substack{\delta_i=o \text{ or } 1 \\ i = 1,2,\ldots,k}} \{m;x_1-\delta_1,\ldots,x_k-\delta_k\} .$$

A little consideration now shows that we can assume that $\sum_i (x_i-\delta_i) \leqq m$ and apply the induction hypothesis to obtain

$$\{m+1;x_1,x_2,\ldots,x_k\} = \sum_{\substack{\delta_i=o \text{ or } 1 \\ i=1,2,\ldots,k}} \prod_{i=1}^{k} \binom{m+1}{x_i-\delta_i} \left[1 - \frac{\sum_i (x_i-\delta_i)}{m+1}\right] .$$

The finite sum on the right-hand side of this equation is now broken into two sums each containing 2^{k-1} terms, the first sum corresponding to $\delta_1 = 0$ and the second corresponding to $\delta_1 = 1$.

$$\{m+1;x_1,x_2,\ldots,x_k\} = \sum_{\substack{\delta_i = o \text{ or } 1 \\ i=2,\ldots,k}} \prod_{i=2}^{k} \binom{m+1}{x_i-\delta_i} \left[\left\{ 1 - \frac{\sum_{i=2}^{k}(x_i-\delta_i)}{m+1} \right\} \binom{m+1}{x_1} - \frac{x_1}{m+1} \cdot \binom{m+1}{x_1} \right]$$

$$+ \sum_{\substack{\delta_i = o \text{ or } 1 \\ i=2,\ldots,k}} \prod_{i=2}^{k} \binom{m+1}{x_i-\delta_i} \left[\left\{ 1 - \frac{\sum_{i=2}^{k}(x_i-\delta_i)}{m+1} \right\} \binom{m+1}{x_1-1} - \frac{(x_1-1)}{(m+1)} \binom{m+1}{x_1-1} \right]$$

After some simplification we obtain

$$\{m+1;x_1,x_2,\ldots,x_k\} = \binom{m+2}{x_1} \sum_{\substack{\delta_i = o \text{ or } 1 \\ i = 2,\ldots,k}} \prod_{i=2}^{k} \binom{m+1}{x_i-\delta_i} \left[1 - \frac{\sum_{i=2}^{k}(x_i-\delta_i)}{m+1} - \frac{x_1}{m+2} \right] .$$

The argument can now be repeated for $i = 2, \ldots, k$ to complete the proof.

In the notation of section 2.2 and proceeding similarly to (2.2.7) we can establish the general identity

$$(3.2.3) \quad \{n;x_1,\ldots,x^*_{i_1},\ldots,x^*_{i_r},\ldots,x_k\} = \prod_{j=1}^{r} \binom{n+2}{x_{i_j}} \prod_{j\in S} \binom{n+1}{x_j} \left[1 - \frac{s^*}{n+2} - \frac{s}{n+1} \right] .$$

Next, proceeding similarly to Theorem 3.2.1 we can establish the following theorem which is completely analogous to Theorem 2.6.1.

Theorem 3.2.2:

Let n, x_i, δ_i $(i = 1,\ldots,k)$ be non-negative integers. For $n \geq 1$,

consider the function defined as follows:

$$\{0;\, 0,\, 0,\, \ldots,\, 0\}^{-} = 1$$

$$(3.2.4) \qquad \{n;x_1,x_2,\ldots,x_k\}^{-} = \begin{cases} 0 & \text{if } \sum_i x_i > n \\ \displaystyle\sum_{\substack{\delta_i = o \text{ or } 1 \\ i=1,\ldots,k}} (-1)^{\sum_i \delta_i} \{n-1;x_1-\delta_1,\ldots,x_k-\delta_k\}^{-} , & \text{otherwise.} \end{cases}$$

Then, with $\sum_i x_i \leqq n$, we have

$$(3.2.5) \qquad \{n;x_1,\ldots,x_k\}^{-} = (-1)^{\sum_i x_i} \prod_{i=1}^{k} \binom{n+1}{x_i} \left[1 - \frac{\sum_i x_i}{n+1}\right] .$$

We will now consider two more recursions. In the first case the recursion is suggested by (3.2.1) but we are not able to give an explicit expression. In the second case the recursion is suggested by (2.2.4) and we are also able to give an explicit expression but it is not essentially analogous to the multinomial.

For $n \geqq 1$, consider then the function $\{n;x_1,\ldots,x_k\}_1$ defined by the following relation:

$$\{0;\, 0,\, 0,\, \ldots,\, 0\}_1 = 1$$

$$(3.2.6) \qquad \{n;x_1,x_2,\ldots,x_k\}_1 = \begin{cases} 0 & \text{if } \sum_i x_i > n \\ \displaystyle\sum_{\substack{\delta_i = o \text{ or } 1 \text{ or } 2 \\ i = 1,2,\ldots,k}} \{n-1;x_1-\delta_1,x_2-\delta_2,\ldots,x_k-\delta_k\}_1 & \text{otherwise} \end{cases}$$

We first remark that in the second member on the right-hand side of (3.2.6), 'Σ' stands for the sum of 3^k terms. As remarked earlier, we could not find an explicit expression for $\{n;x_1,\ldots,x_k\}_1$ but it is easy to show that $\{n;x_1,\ldots,x_k\}_1$ gives the number of vectors in the appropriate subset $[n;x_1,\ldots,x_k]_1$ (with usual congruence properties) of the set of vectors of non-negative integers $(a_1,\ldots,a_n)$ satisfying

(3.2.7)
- (i) $0 \leqq a_1 \leqq a_2 \leqq \cdots \leqq a_n$
- (ii) $0 \leqq a_i \leqq ki \qquad i = 1, 2, \ldots, n$
- (iii) If $a_r (r < n)$ is the first positive element, then the equality in (i) is possible only for at most two elements at a time.

Illustration: For $k = 2$, $n = 3$, the vectors in the set defined in (3.2.7) are

000,001,002,003,004,005,006,011,012,013,014,015,
016,022,023,024,025,026,033,034,035,036,044,045,
046,112,113,114,115,116,122,123,124,125,126,133,
134,135,136,144,145,146,223,224,225,226,233,234,
235,236,244,245,246.

The following "triangles" give values of $\{n;x_1,x_2\}_1$ for $n = 0, 1, 2, 3, 4, 5$. The entries in the first row correspond to values of x_1, x_2 such that $x_1 + x_2 = 0$, those in the second correspond to values of x_1, x_2 such that $x_1 + x_2 = 1$, those in the third correspond to values of x_1, x_2 such that $x_1 + x_2 = 2$ and so on.

n = 0

1

n = 1

1

1 1

n = 2

1

2 2

2 3 2

n = 3

1

3 3

5 8 5

4 10 10 4

n = 4

1

4 4

9 15 9

12 30 30 12

9 30 45 30 9

n = 5

1

5 5

14 24 14

25 63 63 25

30 105 147 105 30

21 114 147 147 114 21

Next we consider the function defined by the following recursion, for $n \geqq 1$,

$$(0;\ 0,\ 0,\ \ldots,\ 0)_m = 1$$

$$(3.2.8) \quad (n;x_1,x_2,\ldots,x_k)_m = \begin{cases} 0 & \text{if } \sum_i x_i > n \\ \sum_{j=o}^{\min(x_1,\ldots,x_k)} (n-1;x_1-j,x_2-j,\ldots,x_k-j)_m , & \text{otherwise .} \end{cases}$$

Here $\min(x_1,x_2,\ldots,x_k)$ stands for the minimum of the integers $x_1,x_2,\ldots,x_k$. Recalling that n, x_i $(i = 1,\ldots,n)$ are non-negative integers and the convention adopted throughout that $(n;x_1,\ldots,x_k)_m$ vanishes when any of the x_i's is negative, we observe first that

$$(3.2.9) \quad (n;x_1,x_2,\ldots,x_k)_m = 0 \qquad \text{if } x_1 \neq x_2 \neq \ldots \neq x_k .$$

Let $x_1 = x_2 = \ldots = x_k = x$; then (3.2.8) can be rewritten for $n \geqq 1$ as follows:

$$(3.2.10) \quad (n;x,\ldots,x)_m = \begin{cases} 0 & \text{if } kx > n \\ \sum_{j=o}^{x} (n-1;x-j,\ldots,x-j)_m & \text{otherwise .} \end{cases}$$

We remark that, for $kx \leqq n$ $(k \geqq 1)$ if $n \equiv 0 \pmod k$ then the maximum value that x can assume is $[\frac{n}{k}] = \frac{n}{k}$. But if $n \equiv v \pmod k$, $v = 1,2,\ldots,k-1$ then the maximum value that x can take is $[\frac{n-v}{k}]$. In either case equation (3.2.11) gives an explicit expression

for $(n;x, \ldots, x)_m$ as can be easily verified, namely,

$$(n;x,x,\ldots,x)_m = \binom{n+x}{x}\left[1 - \frac{kx}{n+1}\right] \qquad (kx \leqq n) . \tag{3.2.11}$$

It is debatable if $(n;x_1,\ldots,x_k)_m$ can strictly be termed as an analogue of the multinomial theorem. Moreover, a closer look at the expression for $(n;x,\ldots,x)_m$ shows that the two functions $(n;x_1,\ldots,x_k)$ and $(n;x,x,\ldots,x)_m$ are somewhat related. However, we shall not enter into these combinatorial details which seem of little interest.

We finally remark that if we replace $\min(x_1,x_2,\ldots,x_k)$ by the $\max(x_1,x_2,\ldots,x_k)$ on the right-hand side of (3.2.8) we end up again with (3.2.10). This is obvious in view of our remarks following (3.2.8).

No other relation of the type discussed above seems to give explicit expression. It would be interesting to know if there exist some other non-trivial functions of the type we have discussed, i.e. which can be defined in terms of a recurrence relation and have explicit expressions.

§ 3.3 Some applications

In this section we will first show the connection between $\{n;x_1,\ldots,x_k\}$ and a certain set of vectors and then give some applications of $\{n;x_1,\ldots,x_k\}$. The results obtained here are analogous to those obtained in section 2.6.

Consider then the set $B_{n,k}$ of vectors of non-negative integers $(b_1,b_2,\ldots,b_k)$ whose elements satisfy

$$
(3.3.1) \quad \begin{cases} \text{(i)} & 0 \leqq b_1 \leqq b_2 \leqq \dots \leqq b_n \\ \text{(ii)} & 0 \leqq b_i \leqq ki \qquad i = 1, 2, \dots, n \\ \text{(iii)} & \text{If } b_r \ (r < n) \text{ is the first positive} \\ & \text{element then the equality in (i) is} \\ & \text{impossible.} \end{cases}
$$

We remark that $B_{n,k}$ is a subset of $A_{n,k}$ of section 2.3, as is to be expected.

Let us now introduce the subset $S\{n;x_1,\dots,x_k\}$ of $B_{n,k}$ such that every vector in $S\{n;x_1,\dots,x_k\}$ has exactly x_i of its <u>positive</u> elements $\equiv i \pmod k$, $i = 1, 2, \dots, k$ respectively. Then the following theorem establishes the connection between the number of elements in $S\{n;x_1,\dots,x_k\}$ and $\{n;x_1,\dots,x_k\}$ of equation (3.2.2).

Theorem 3.3.1:

The number of vectors in $S\{n;x_1,\dots,x_k\}$ is given by $\{n;x_1,\dots,x_k\}$ of equation (3.2.2).

Outline of Proof:

The proof consists in establishing a one-one correspondence between the elements of $S\{n;x_1,\dots,x_k\}$ and $T = \bigcup_{\substack{\delta_i = 0 \text{ or } 1 \\ i=1,\dots,k}} S\{n-1;x_1-\delta_1,\dots,x_k-\delta_k\}$.

It is easily shown that the mapping P defined in the proof of Lemma 2.3.1

is one-one from $S\{n;x_1,\ldots,x_k\}$ onto T . This completes the proof.

Denote by $B^*_{n,k}$ the number of elements in $B_{n,k}$. Then proceeding analogously to section 2.5 we have

$$B^*_{n,k} = \sum_{c=0}^{n} \left(1 - \frac{c}{n+1}\right) \binom{(n+1)k}{c} . \tag{3.3.2}$$

Finally, we state the following corollaries of Theorem 3.3.1. The proofs are straightforward in view of our discussion in section 2.6 and will be omitted.

Corollary 1:

Consider the set of vectors of non-negative integers $(b_1,b_2,\ldots,b_n)$ such that

$$\left\{\begin{array}{ll} \text{(i)} & 0 \leqq b_1 \leqq b_2 \leqq \cdots \leqq b_n \\ \text{(ii)} & b_i \leq ki + t \quad i=1,2,\ldots,n;\ t \text{ is an integer, } 0 \leqq t \leqq k . \\ \text{(iii)} & \text{If } b_r\ (r < n) \text{ is the first positive} \\ & \text{element then equality in (i) is impossible.} \end{array}\right. \tag{3.3.3}$$

Letting $S_t\{n;x_1,\ldots,x_k\}$ denote the appropriate subset of (3.3.3) with the usual congruence properties, and $S^*_t\{n;x_1,x_2,\ldots,x_k\}$ the number of elements in $S_t\{n;x_1,\ldots,x_k\}$ we have

$$S^*_t\{n;x_1,x_2,\ldots,x_k\} = \{n;x_{t+1},\ldots,x_k,x^*_1,\ldots,x^*_t\} \tag{3.3.4}$$

$$= \prod_{i=1}^{t} \binom{n+2}{x_i} \prod_{j=t+1}^{k} \binom{n+1}{x_j} \left[1 - \frac{\Sigma x_i}{n+2} - \frac{\Sigma x_j}{n+1}\right] .$$

Corollary 2:

Let $B_n(a,b)$ denote the set of vectors of non-negative integers $(b_1,\ldots,b_n)$ satisfying

$$(3.3.5)\quad \begin{cases} \text{(i)} & 0 \le b_1 \le b_2 \le \cdots \le b_n \\ \text{(ii)} & 0 \le b_i \le a + (i-1)b \quad i=1,2,\ldots,n; a,b \text{ are integers } \ge 0 \\ \text{(iii)} & \text{If } b_r \ (r < n) \text{ is the first positive element} \\ & \text{then equality in (i) is impossible.} \end{cases}$$

Then the number of elements in $B_n(a,b)$ is given by

$$(3.3.6)\qquad B_n^*(a,b) = \sum_{s=0}^{n} \binom{bn+a}{s}\left[1 - \frac{bs}{bn+a}\right] .$$

Corollary 3:

In (3.3.1) if we let $r = 1$, then we have b_1 as the first positive element and accordingly the conditions (3.3.1) reduce to

$$(3.3.7)\qquad \begin{cases} 0 < b_1 < b_2 < \cdots < b_n \\ 0 < b_i \le ki \qquad i = 1, 2, \ldots, n \end{cases}$$

Let $B(n)$ be the set of vectors with positive integral elements $(b_1,b_2,\ldots,b_n)$ satisfying (3.3.7). Let $B\{n;x_1,\ldots,x_k\}$ with $\sum_i x_i = n$, denote the subset of $B(n)$ such that every vector in $B\{n;x_1,\ldots,x_k\}$ has exactly x_i of its elements $\equiv i \pmod k$, $i = 1, 2, \ldots, k$ respectively. Then a little consideration shows that the number of elements in $B\{n;x_1,\ldots,x_k\}$ is

given by

$$(3.3.8) \qquad B^*\{n;x_1,\ldots,x_k\} = \begin{cases} \frac{1}{n+1} \prod_{i=1}^{k} \binom{n+1}{x_i} , & \sum_i x_i = n , \\ 0 & \text{otherwise,} \end{cases}$$

and the number of elements in $B(n)$ is given by

$$(3.3.9) \qquad B^*(n) = \frac{1}{n+1} \binom{(n+1)k}{n} .$$

We shall see that these results will help us solve a complicated difference equation in the next section.

Corollary 4:

Consider the set of vectors of non-negative integers whose elements $(b_1,\ldots,b_n)$ satisfy

$$(3.3.10) \qquad \begin{cases} \text{(i)} & 0 \leq b_1 < b_2 < \ldots < b_n \\ \text{(ii)} & b_i \leq ki \qquad i = 1, 2, \ldots, n \end{cases}$$

If $B_1\{n;x_1,\ldots,x_k\}$ denotes the usual subset of the set of vectors defined by (3.3.10) then the number of elements in $B_1\{n;x_1,\ldots,x_k\}$ is given by

$$(3.3.11) \qquad B_1^*\{n;x_1,\ldots,x_k\} = \begin{cases} \frac{3}{n+2} \prod_{i=1}^{k} \binom{n+2}{x_i} & \text{if } \sum x_i = n-1 \\ \frac{1}{n+1} \prod_{i=1}^{k} \binom{n+1}{x_i} & \text{if } \sum x_i = n \\ 0 & \text{otherwise} \end{cases} .$$

§ 3.4 A difference equation

In this section we consider a partial difference equation of a somewhat complicated nature and solve it explicitly. We will only consider the case $k=2$ which is of special interest and this is also the case in which we have been able to give an explicit solution. In the general case an explicit solution seems difficult to obtain.

Let n, e, x_i $(i=1,2)$ with $x_1+x_2 = n$ be non-negative integers. For $n \geqq 1$ consider the difference equation defined as follows:

$$B^*(0, 0, ; 0) = 1 ,$$

$$B^*(n, e; x_1) = 0 \quad \text{if } e < n, \ e > 2n$$

$$B^*(n, e; x_1) = \begin{cases} \sum_{j=1}^{e-1} B^*(n-1, j; x_1-1) & \text{if } e \text{ is odd} \\ \sum_{j=1}^{e-1} B^*(n-1, j; x_1) & \text{if } e \text{ is even .} \end{cases} \tag{3.4.1}$$

We will soon see that this difference equation arises as a result of the relationships between certain subsets of $B(n)$ (with $k = 2$) defined in Corollary 3 section 3.3. For the moment, consider the subset $B(n,e)$ of $B(n)$ such that the last element of every vector in $B(n,e)$ equals e . Denoting by S^* the number of elements in S , we have, for $n \geqq 1$, $B^*(n,n) = 1$. This follows because $b_n=n$ and the only vector with $b_n=n$ is $(1,2,\ldots,n)$. Let $B^*(0,0) = 1$.

Illustration: $B(3)$ consists of the following vectors

123,124,125,126,134,135,136,145,146,

234,235,236,245,246.

$B^*(3) = 14$ and $B(3,4) = [124,134,234]$, $B(3,6) = [126,136,146,236,246]$, etc.

Then we have the

Theorem 3.4.1:

$$B^*(n,e) = \begin{cases} 0 & \text{if } e < n \text{ and } e > 2n \\ \sum_{j=1}^{e-1} B^*(n-1,j) & \text{if } n \leqq e \leqq 2n . \end{cases} \tag{3.4.2}$$

Indication of Proof:

$B^*(n,e) = 0$ for $e < n$ and $e > 2n$ by <u>definition</u>. For $n \leqq e \leqq 2n$, associate with every vector B_n in $B(n,e)$ a vector of $(n-1)$ elements, $S(B_n)$ obtained by suppressing the last element of B_n . Then a little consideration shows that S defines a one-one mapping from $B(n,e)$ <u>onto</u> $\bigcup_{j=1}^{e-1} B(n-1,j)$.

It is now easily verified that $B^*(n,e)$ is explicitly given by

$$B^*(n,e) = \frac{2n+1-e}{n+1} \binom{e}{n} . \tag{3.4.3}$$

Let us now partition the set $B(n,e)$ into subsets $B(n,e;x_1)$ such that every vector in $B(n,e;x_1)$ has exactly x_1 elements $\equiv 1 \pmod 2$.

We wish to emphasize that the set $B(n,e;x_1)$ should not be confused with the set $B\{n;x_1,x_2\}$ of Corollary 3 section 3.3. Recall that $B\{n;x_1,x_2\}$ is the subset of $B(n)$ such that every vector in $B\{n;x_1,x_2\}$ has exactly x_i of its elements $\equiv i \pmod k$ $i = 1,2,$ respectively. We further note that $B(n,e;x_1)$ is a subset of $B\{n;x_1,x_2\}$. (This is not true in general.) Then it is easy to see that

$$\sum_{e=n}^{2n} B^*(n,e;x_1) = \frac{1}{n+1}\binom{n+1}{x_1}\binom{n+1}{x_1+1} . \tag{3.4.4}$$

Illustration: In the following tables the row sums give $B^*\{n;x_1,n-x_1\}$ and the column sums give $B^*(n,e)$. The entry in the x_1 th row $(x_1=0,1,\ldots,n)$ and the e^{th} column $(e=n,n+1,\ldots,2n)$ gives $B^*(n,e;x_1)$.

n=2

x_1 \ e	2	3	4	$B^*\{2;x_1,2-x_1\}$
0			1	1
1	1	1	1	3
2		1		1
$B^*(2,e)$	1	2	2	5

n=3

x_1 \ e	3	4	5	6	$B^*\{3;x_1,3-x_1\}$
0				1	1
1		2	1	3	6
2	1	1	3	1	6
3			1		1
$B^*(3,e)$	1	3	5	5	14

n=4

x_1 \ e	4	5	6	7	8	$B^*(4;x_1,4-x_1)$
0					1	1
1			3	1	6	10
2	1	2	5	6	6	20
3		2	1	6	1	10
4				1		1
$B^*(4,e)$	1	4	9	14	14	42

n=5

x_1 \ e	5	6	7	8	9	10	$B^*(5;x_1 5-x_1)$
0						1	1
1				4	1	10	15
2		3	3	14	10	20	50
3	1	2	8	9	20	10	50
4			3	1	10	1	15
5					1		1
$B^*(5,e)$	1	5	14	28	42	42	232

	$x_1 \backslash e$	6	7	8	9	10	11	12		
	0							1	1	
	1					5	1	15	21	
	2			6	4	30	15	50	105	
n=6	3	1	3	11	20	40	50	50	175	$B^*\{6;x_1,6-x_1\}$
	4		3	3	20	14	50	15	105	
	5				4	1	15	1	21	
	6						1		1	
		1	6	20	48	90	132	132	429	

$B^*(6,e)$

A little consideration now shows that $B^*(n,e;x_1)$, the number of elements in $B(n,e;x_1)$ satisfies the difference equation (3.4.1). Furthermore, we have

$$(3.4.5) \qquad B^*(n,2n;x_1) = B^*\{n-1;x_1,n-x_1-1\} = \frac{\binom{n}{x_1}\binom{n}{x_1+1}}{n}$$

and

$$(3.4.6) \qquad B^*(n,2n-1;x_1) = B^*\{n-1;x_1-1,n-x_1\} = \frac{\binom{n}{x_1-1}\binom{n}{x_1}}{n} .$$

Equations (3.4.1), (3.4.5) and (3.4.6) determine $B^*(n,e;x_1)$ for all values of $e(n \leqq e \leqq 2n)$. Indeed, if e is odd and equals $2n-(2v+1)$, say, with $v=0,1,\dots,[\frac{n-1}{2}]$, we can show by induction that

$$(3.4.7)\qquad B^*(n,2n-2v-1;x_1) = (v+1)\,\frac{\binom{n-v}{x_1}\binom{n-v}{x_1-v-1}}{n-v}$$

and if e is even and equals $2n-2v$, say, with $v = 0, 1, \ldots, [\frac{n}{2}]$ then

$$(3.4.8)\qquad B^*(n,2n-2v;x_1) = v\,\frac{\binom{n-v}{x_1}\binom{n-v}{x_1-v}}{n-v} + (v+1)\,\frac{\binom{n-v}{x_1+1}\binom{n-v}{x_1-v}}{n-v} .$$

Equations (3.4.7) and (3.4.8) give an explicit solution of the difference equation (3.4.1). We remark that a difference equation may be set up for the general case $k > 2$ in a similar fashion.

§ 3.5 Miscellany

It is only fitting that we wind up our discussion with ballot theorems. We recall that in Chapter I, and again in Chapter II, we gave an explicit expression for the number of lattice paths to any point below the line $x = ky$ when $k > 0$ is an integer. Our object is to obtain an explicit expression for the number of paths to any point when $k = \frac{1}{\mu}$, $\mu > 1$ is an integer. It may however be noted that although Takács' solution of the ballot theorem covers this case, his method does not easily yield an explicit expression. Similarly, by measuring the distance a_i of the path parallel to the y-axis and using our approach in Chapter II we can not only obtain an expression for the number of paths to $(n,n\mu-1)$ lying below the line $y = \mu x$ but also classify the paths according to the congruence properties of the elements of their path vectors. It is however not clear as to how to use this approach to give the number of paths to any point in the plane (lying below $y = \mu x$).

The method that we use here has been suggested by J. Sarangi. We will only consider the case $\mu = 2$. The argument for any $\mu \geq 3$ is similar. Consider first the special case of points $(n,2n)$. For $i < n$, $[\frac{2n-1}{n} i] = 2i-1$ and hence the number of paths from $(0,0)$ to $(n,2n)$ below the line $y = 2x$ is the same as the number of paths from $(0,0)$ to $(n,2n-1)$ below the line joining $(0,0)$ to $(n,2n-1)$. Since $(n,2n-1) = 1$, it follows from a theorem proved in [4] that the required number of paths is given by

$$(3.5.1) \qquad (n,2n)^* = \frac{1}{3n-1} \binom{3n-1}{n} .$$

Let us next consider paths to points $(n,2m)$ with $m < n$. There are $\binom{n+2m}{n}$ paths to this point in all. Our method consists of counting all paths that intersect the line $y = 2x$. A path from $(0,0)$ to $(n,2m)$ can intersect the line $y = 2x$ for the <u>first time</u> at a point whose ordinate is i where $i = 1,2,\ldots,2m$. If it intersects $y = 2x$ for the first time at a lattice point $(i,2i)$ then there are $\frac{2}{3i-1}\binom{3i-1}{i}$ such paths upto $(i,2i)$ and $\binom{n+2m-3i}{2m-2i}$ paths from $(i,2i)$ to $(n,2m)$. Thus the total number of paths intersecting the line $y = 2x$ for the first time at a lattice point is

$$\sum_{i=1}^{m} \frac{2}{3i-1} \binom{3i-1}{i} \binom{n+2m-3i}{2m-2i} .$$

Also the paths intersecting the line $y = 2x$ for the first time at a point with ordinate $(2i+1)$ must travel above the line $y = 2x$ upto the point $(i,2i+1)$ and then to $(i+1,2i+1)$ and from $(i+1,2i+1)$ to

$(n,2m)$. The number of paths to $(i,2i+1)$ which lie entirely above the line $y = 2x$ is the same as the number of paths to $(2i+1,i)$ which lie entirely below the line $x = 2y$. The total number of such paths is $\frac{1}{3i+1}\binom{3i+1}{i}$ and hence the total number of paths from $(0,0)$ to $(n,2m)$ that intersect the line $y = 2x$ for the first time at points with ordinate $(2i+1)$ is $\sum_{i=0}^{m-1} \frac{1}{3i+1}\binom{3i+1}{i}\binom{n+2m-3i-2}{n-i-1}$.

Thus the total number of paths from $(0,0)$ to $(n,2m)$ that lie entirely below the line $y = 2x$ is given by

$$(3.5.2) \qquad (n,2m)^* = \binom{n+2m}{n} - \sum_{i=1}^{m} \frac{2}{3i-1}\binom{3i-1}{i}\binom{n+2m-3i}{n-i}$$

$$- \sum_{i=0}^{m-1} \frac{1}{3i+1}\binom{3i+1}{i}\binom{n+2m-3i-2}{n-i-1}$$

A rather similar argument shows that the total number of paths from $(0,0)$ to $(n,2m+1)$ that lie entirely below the line $y = 2x$ is given by

$$(3.5.3) \qquad (n,2m+1)^* = \binom{n+2m+1}{n} - \sum_{i=1}^{m} \frac{2}{3i-1}\binom{3i-1}{i}\binom{n+2m+1-3i}{n-i}$$

$$- \sum_{i=0}^{m-1} \frac{1}{3i+1}\binom{3i+1}{i}\binom{n+2m-3i-1}{n-i-1} .$$

The generalisation to any $\mu \geq 3$ is now straightforward. Indeed, we have

$$(3.5.4) \qquad (n,n\mu)^* = \frac{1}{n(\mu+1)-1}\binom{n(\mu+1)-1}{n}$$

and if $0 \leqq m < n-1$ and $0 < j \leqq \mu-1$ then

$$(3.5.5) \quad (n,m\mu)^* = \binom{n+m\mu}{n} - \sum_{i=1}^{m} \frac{2}{i(\mu+1)-1} \binom{i(\mu+1)-1}{i}\binom{n+m\mu-i(\mu+1)}{n-i}$$

$$- \sum_{i=o}^{m-1} \sum_{v=1}^{\mu-1} \frac{1}{(\mu+1)i+v} \binom{(\mu+1)i+v}{i} \binom{n+m\mu-(\mu+1)i-v-1}{n-i-1}$$

and

$$(3.5.6) \quad (n,m\mu+j)^* = \binom{n+m\mu+j}{n} - \sum_{i=1}^{m} \frac{2}{i(\mu+1)-1} \binom{i(\mu+1)-1}{i}\binom{n+\mu m+j-i(\mu+1)}{n-i}$$

$$- \sum_{i=o}^{m-1} \sum_{v=1}^{\mu-1} \frac{1}{(\mu+1)i+v} \binom{(\mu+1)i+v}{i} \binom{n+m\mu+j-(\mu+1)i-v-1}{n-i-1} .$$

Finally, we wish to establish the identity

$$(3.5.7) \quad \sum_{i=o}^{n} \frac{1}{i+1} \binom{2i}{i} \binom{2n-2i}{n-i} = \binom{2n+1}{n}$$

using the ballot theorem. The right-hand side of (3.5.7) gives the total number of lattice paths from $(0,0)$ to $(n+1,n)$. To complete the proof we observe that any path from $(0,0)$ to $(n+1,n)$ must either touch or cross the line $x = y$ for the <u>last time</u> at some lattice point (including $(0,0)$) and from there on the path must lie entirely below the line $x = y$. Thus if L_i denotes the number of paths from $(0,0)$ to $(n+1,n)$ which touch

or cross the line $x = y$ for the last time at the point $(n-i,n-i)$, $i = 0, 1, \ldots, n$ then clearly

$$(3.5.8) \qquad L_i = \binom{2n-2i}{n-i} \cdot \frac{1}{i+1} \binom{2i}{i}$$

and $\sum_{i=o}^{n} L_i$ gives the total number of paths to $(n+1,n)$. This proves our assertion.

We remark that the identity (3.5.7) was recently posed as an advanced problem in the American Mathematical Monthly [14]. The argument used here is essentially due to Feller who in [5] has made very elegant applications of this and similar ideas to solve much more difficult problems.

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